

SYMPLECTIC REFLECTION ALGEBRAS

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ABSTRACT. These lecture notes are based on an introductory course given by the author at the summer school “Noncommutative Algebraic Geometry” at MSRI in June 2012. The emphasis throughout is on examples to illustrate the many different facets of symplectic reflection algebras. Exercises are included at the end of each lecture in order for the student to get a better feel for these algebras.

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INTRODUCTION

The purpose of these notes is to give the reader a flavor of, and basic grounding in, the theory of symplectic reflection algebras. These algebras, which were introduced by Etingof and Ginzburg in [24], are related to an astonishingly large number of apparently disparate areas of mathematics such as combinatorics, integrable systems, real algebraic geometry, quiver varieties, resolutions of symplectic singularities and, of course, representation theory. As such, their exploration entails a journey through a beautiful and exciting landscape of mathematical constructions. In particular, as we hope to illustrate throughout these notes, studying symplectic reflection algebras involves a deep interplay between geometry and representation theory.

A brief outline of the content of each lecture is as follows. In the first lecture we motivate the definition of symplectic reflection algebras by considering deformations of certain quotient singularities. Once the definition is given, we state the Poincaré-Birkhoff-Witt theorem, which is of fundamental importance in the theory of symplectic reflection algebras. This is the first of many

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analogies between Lie theory and symplectic reflection algebras. We also introduce a special class of symplectic reflection algebras, the *rational Cherednik algebras*. This class of algebras gives us many interesting examples of symplectic reflection algebras that we can begin to play with. We end the lecture by describing the double centralizer theorem, which allows us to relate the symplectic reflection algebra with its spherical subalgebra, and also by describing the centre of these algebras.

In the second lecture, we consider symplectic reflection algebras at $t = 1$. We focus mainly on rational Cherednik algebras and, in particular, on category \mathcal{O} for these algebras. This category of finitely generated $H_c(W)$ -modules has a rich, combinatorial representation theory and good homological properties. We show that it is a highest weight category with finitely many simple objects.

Our understanding of category \mathcal{O} is most complete when the corresponding complex reflection group is the symmetric group. In the third chapter we study this case in greater detail. It is explained how results of Rouquier, Vasserot-Varangolo and Leclerc-Thibon allow us to express the multiplicities of simple modules inside standard modules in terms of the canonical basis of the “level one” Fock space for the quantum affine Lie algebra of type A . A corollary of this result is a character formula for the simple modules in category \mathcal{O} . We end the lecture by stating Yvonne’s conjecture which explains how the above mentioned result should be extended to the case where W is the wreath product $\mathfrak{S}_n \wr \mathbb{Z}_m$.

The fourth lecture deals with the Knizhnik-Zamolodchikov (KZ) functor. This remarkable functor allows one to relate category \mathcal{O} to modules over the corresponding cyclotomic Hecke algebra. In fact, it is an example of a quasi-hereditary cover, as introduced by Rouquier. The basic properties of the functor are described and we illustrate these properties by calculating explicitly what happens in the rank one case.

The final lecture deals with symplectic reflection algebras at $t = 0$. For these parameters, the algebras are finite modules over their centres. We explain how the geometry of the centre is related to the representation theory of the algebras. We also describe the Poisson structure on the centre and explain its relevance to representation theory. For rational Cherednik algebras, we briefly explain how one can use the notion of Calogero-Moser partitions, as introduced by Gordon and Martino, in order to decide when the centre of these algebras is regular.

There are several very good lecture notes and survey papers on symplectic reflection algebras and rational Cherednik algebras. For instance [23], [22], [30], [31] and [44]. I would also strongly suggest to anyone interested in learning about symplectic reflection algebras to read the original paper [24] by P. Etingof and V. Ginzburg, where symplectic reflection algebras were first defined¹. It makes a great introduction to the subject and is jam packed with ideas and clever arguments. As noted briefly above, there are strong connections between symplectic reflection algebras and several other areas of mathematics. Due to lack of time and energy, we haven’t touched upon those connections here. The interested reader should consult one of the surveys mentioned above. A final

¹A few years after the publication of [24] it transpired that the definition of symplectic reflection algebras had already appeared in a short paper [19] written by V. Drinfeld in the eighties.

remark: to make the lectures as readable as possible, there is only a light sprinkling of references in the body of the text. Detailed references can be found at the end of each lecture.

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1. SYMPLECTIC REFLECTION ALGEBRAS

1.1. Motivation. Let V be a finite dimensional vector space over \mathbb{C} and $G \subset GL(V)$ a finite group. Fix $\dim V = m$. It is a classical problem in algebraic geometry to try and understand the space $V/G = \text{Spec } \mathbb{C}[V]^G$, see Wemyss' lectures [54]. At the most basic level, we would like to try and answer the questions

Question 1.1. Is the space V/G singular?

Or, more generally

Question 1.2. How singular is V/G ?

The answer to the first question is a classical theorem due to Chevalley and Shephard-Todd. Before I can state their theorem, we need the notion of a complex reflection, which generalizes the classical definition of reflection encountered in Euclidean geometry.

Definition 1.3. An element $s \in G$ is said to be a *complex reflection* if $\text{rk}(1 - s) = 1$. Then G is said to be a *complex reflection group* if G is generated by S , the set of all complex reflections contained in G .

Combining the results of Chevalley and Shephard-Todd, we have:

Theorem 1.4. *The space V/G is smooth if and only if G is a complex reflection group. If V/G is smooth then it is isomorphic to \mathbb{A}^m , where $\dim V = m$.*

A complex reflection group G is said to be *irreducible* if the reflection representation V is a simple G -module. It is an easy exercise to show that if $V = V_1 \oplus \cdots \oplus V_k$ is the decomposition of V into simple G -modules, then $G = G_1 \times \cdots \times G_k$, where G_i acts trivially on V_j for all $j \neq i$ and (G_i, V_i) is an irreducible complex reflection group. The irreducible complex reflection groups have been classified by Shephard and Todd, [47].

Example 1.5. Let \mathfrak{S}_n the symmetric group act on \mathbb{C}^n by permuting the coordinates. Then, the reflections in \mathfrak{S}_n are exactly the transpositions and hence \mathfrak{S}_n is a complex reflection group. If $\mathbb{C}^n = \text{Spec } \mathbb{C}[x_1, \dots, x_n]$, then the ring of invariants $\mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ is a polynomial ring with generators e_1, \dots, e_n , where

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

is the i th elementary symmetric polynomial. Notice however that $(\mathfrak{S}_n, \mathbb{C}^n)$ is not an irreducible complex reflection group.

Example 1.6 (Non-example). Take $m = 2$ i.e. $V = \mathbb{C}^2$ and G a finite subgroup of $SL_2(\mathbb{C})$. Then it is easy to see that $S = \emptyset$ so G cannot be a complex reflection group. The singular space \mathbb{C}^2/G is called a Kleinian (or Du Val) singularity. The groups G are classified by simply laced Dynkin diagrams i.e. those diagrams of type ADE, and the singularity \mathbb{C}^2/G is an isolated hypersurface singularity in \mathbb{C}^3 .

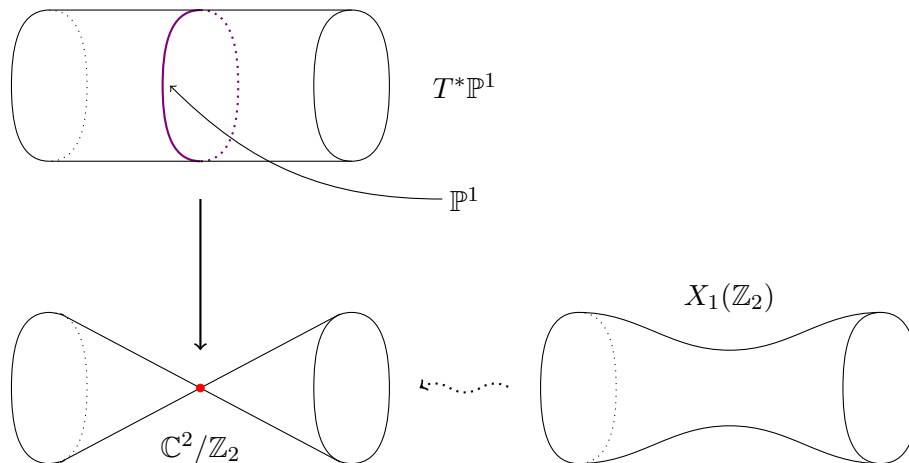


FIGURE 1. The resolution and the deformation of the \mathbb{Z}_2 quotient singularity.

The previous non-example is part of a large class of groups called symplectic reflection groups. This is the class of groups for which one can try to understand V/G using symplectic reflection algebras. If (V, ω) is a symplectic vector space and $G \subset Sp(V)$ then G cannot contain any reflections, thus, it is never a complex reflection group. However, one can define $s \in G$ to be a *symplectic reflection* if $\text{rk}(1 - s) = 2$. The idea here being that a symplectic reflection is the nearest thing to a genuine complex reflection that one can hope for in a subgroup of $Sp(V)$.

Definition 1.7. The triple (V, ω, G) is a *symplectic reflection group* if (V, ω) is a symplectic vector space and $G \subset Sp(V)$ is a finite group that is generated by S , the set of all symplectic reflections in G .

Since a symplectic reflection group (V, ω, G) is not “too far” from being a complex reflection group, one might expect V/G to be “not too singular”. One way to make this precise is to ask whether V/G admits what’s called a crepant resolution (the actual definition of a crepant resolution won’t be important to us in this course). This is indeed the case for many (but not all!) symplectic reflection groups². In order to classify for those groups G for which the space V/G admits a crepant resolution, the first key idea is to try to understand V/G by looking at deformations of the space i.e. some affine variety $\pi : X \rightarrow \mathbb{C}^k$ such that $\pi^{-1}(0) \simeq V/G$ and the map π is *flat*. Intuitively, this is asking that the dimension of the fibers of π don’t change. Then it is reasonable to hope that a generic fiber of π is easier to describe, but still tells us something about the geometry of V/G .

However there is a fundamental problem with this idea. We cannot hope to be able to write down generators and relations for the ring $\mathbb{C}[V]^G$ in general. So it seems like a hopeless task to try and write down deformations of the ring. The second key idea is to try and overcome this problem by introduce non-commutative geometry into the picture. In our case, the relevant non-commutative algebra is the skew group ring.

²Skip to the end of the final lecture for a precise statement.

Definition 1.8. The *skew group ring* $\mathbb{C}[V] \rtimes G$ is, as a vector space, equal to $\mathbb{C}[V] \otimes \mathbb{C}G$ and the multiplication is given by

$$g \cdot f = {}^g f \cdot g, \quad \forall f \in \mathbb{C}[V], g \in G,$$

where ${}^g f(v) = f(g^{-1}v)$ for $v \in V$.

Exercise 1.9. Show that the centre $Z(\mathbb{C}[V] \rtimes G)$ of $\mathbb{C}[V] \rtimes G$ equals $\mathbb{C}[V]^G$.

The above exercise shows that the information of the ring $\mathbb{C}[V]^G$ is encoded in the definition of the skew group ring. On the other hand, the skew group ring has a very explicit, simple presentation. Therefore, we can try to deform $\mathbb{C}[V] \rtimes G$ instead, in the hope that the centre of the deformed algebra is itself a deformation of $\mathbb{C}[V]^G$. We refer the reader to Schedler's lectures [46] for information on the theory of deformations of algebras.

1.2. Symplectic reflection algebras. Thus, symplectic reflection algebras are a particular family of deformations of $\mathbb{C}[V] \rtimes G$, when G is a symplectic reflection group. Fix (V, ω, G) , a symplectic reflection group. Let \mathcal{S} be the set of symplectic reflections in G . For each $s \in \mathcal{S}$, the spaces $\text{Im}(1 - s)$ and $\text{Ker}(1 - s)$ are symplectic subspaces of V with $V = \text{Im}(1 - s) \oplus \text{Ker}(1 - s)$ and $\dim \text{Im}(1 - s) = 2$. We denote by ω_s the 2-form on V whose restriction to $\text{Im}(1 - s)$ is ω and whose restriction to $\text{Ker}(1 - s)$ is zero. Let $\mathbf{c} : \mathcal{S} \rightarrow \mathbb{C}$ be a conjugate invariant function i.e.

$$\mathbf{c}(gsg^{-1}) = \mathbf{c}(s), \quad \forall s \in \mathcal{S}, g \in G.$$

Let TV^* be the tensor algebra on V^* .

Definition 1.10. Let $t \in \mathbb{C}$. The *symplectic reflection algebra* $H_{t,\mathbf{c}}(G)$ is define to be

$$H_{t,\mathbf{c}}(G) = TV^* \rtimes G / \langle [u, v] = t\omega(u, v) - 2 \sum_{s \in \mathcal{S}} \mathbf{c}(s) \omega_s(u, v) \cdot s \mid u, v \in V^* \rangle. \quad (1)$$

Notice that the defining relations of the symplectic reflection algebra are trying to tell you how to commute two vectors in V^* . The expression on the right hand side of (1) belongs to the group algebra $\mathbb{C}G$, so the price for commuting u and v is that one gets an extra term living in $\mathbb{C}G$.

Example 1.11. The simplest non-trivial example is $\mathbb{Z}_2 = \langle s \rangle$ acting on \mathbb{C}^2 . Let $(\mathbb{C}^2)^* = \text{Span}(x, y)$, where $s \cdot x = -x$, $s \cdot y = -y$ and $\omega(y, x) = 1$. Then $H_{t,\mathbf{c}}(\mathfrak{S}_2)$ is the algebra

$$\mathbb{C}\langle x, y, s \rangle / \langle s^2 = 1, sx = -xs, sy = -ys, [y, x] = t - 2\mathbf{c}s \rangle.$$

This example, our “favorite example”, will appear throughout the course.

When t and \mathbf{c} are both zero, we have $H_{0,0}(G) = \mathbb{C}[V] \rtimes G$. If $\lambda \in \mathbb{C}^\times$ then $H_{\lambda t, \lambda \mathbf{c}}(G) \simeq H_{t,\mathbf{c}}(G)$ so we normally only consider the cases $t = 0, 1$.

Example 1.12. Again take $V = \mathbb{C}^2$, then $Sp(V) = SL_2(\mathbb{C})$ so we can take G to be any finite subgroup of $SL_2(\mathbb{C})$. Every $g \neq 1$ in G is a symplectic reflection and $\omega_g = \omega$. Let x, y be a basis of $(\mathbb{C}^2)^*$ such that $\omega(y, x) = 1$. Then

$$H_{t,\mathbf{c}}(G) = \mathbb{C}\langle x, y \rangle \rtimes G / \left\langle [y, x] = t - 2 \sum_{g \in G \setminus \{1\}} \mathbf{c}(g)g \right\rangle.$$

The map

$$(t, \mathbf{c}) \mapsto t - 2 \sum_{g \in G \setminus \{1\}} \mathbf{c}(g)g$$

is an isomorphism $\mathbb{C} \times \mathbb{C}[\mathcal{S}/G] \simeq Z(G)$ of vector spaces, where $Z(G)$ is the centre of the group algebra. Hence the main relation can simply be expressed as $[y, x] = z$ for some (fixed) $z \in Z(G)$. For (many) more properties of the algebras $H_{t, \mathbf{c}}(G)$, see [17] where these algebras were first defined and studied.

There is a natural filtration \mathcal{F} on $H_{t, \mathbf{c}}(G)$, given by putting V^* in degree one and G in degree zero. The crucial result by Etingof and Ginzburg, on which the whole of the theory of symplectic reflection algebras is built, is the Poincaré-Birkhoff-Witt (PBW) Theorem.

Theorem 1.13. *The map $\sigma(v) \mapsto v, \sigma(g) \mapsto g$ defines an isomorphism of algebras*

$$\mathrm{gr}_{\mathcal{F}}(H_{t, \mathbf{c}}(G)) \simeq \mathbb{C}[V] \rtimes G,$$

where $\sigma(D)$ denotes the symbol, or leading term, of $D \in H_{t, \mathbf{c}}(G)$ in $\mathrm{gr}_{\mathcal{F}}(H_{t, \mathbf{c}}(G))$. Equivalently, there is an isomorphism of vector spaces $H_{t, \mathbf{c}}(G) \simeq \mathbb{C}[V] \otimes \mathbb{C}G$.

The key point of the PBW theorem is that it gives us an explicit basis of the symplectic reflection algebra. The proof of this theorem is an application of a general result by Braverman and Gaitsgory [7]. If I is a two-sided ideal of $TV^* \rtimes G$ generated by a space U of (not necessarily homogeneous) elements of degree at most two then [7, Theorem 0.5] gives necessary and sufficient conditions on U so that the quotient $TV^* \rtimes G/I$ has the PBW property. The PBW property immediately implies that $H_{t, \mathbf{c}}(G)$ enjoys some good ring-theoretic properties, for instance:

Corollary 1.14. (1) *The algebra $H_{t, \mathbf{c}}(G)$ is a prime Noetherian ring.*

(2) *$H_{t, \mathbf{c}}(G)$ has finite global dimension (in fact, $\mathrm{gl.dim} H_{t, \mathbf{c}}(G) \leq m$).*

1.3. The rational Cherednik algebra. There is a standard way to construct a large number of symplectic reflection groups - by creating them out of complex reflection groups. This class of symplectic reflection algebras is by far the most important and, thus, have been most intensively studied out of all symplectic reflection algebras. So let W be a complex reflection group, acting on its reflection representation \mathfrak{h} . Then W acts diagonally on $\mathfrak{h} \times \mathfrak{h}^*$. The space $\mathfrak{h} \times \mathfrak{h}^*$ has a natural pairing $(\cdot, \cdot) : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$ defined by $(y, x) = x(y)$, and

$$\omega((y_1, x_1), (y_2, x_2)) := (y_1, x_2) - (y_2, x_1)$$

defines a W -equivariant symplectic form on $\mathfrak{h} \times \mathfrak{h}^*$. One can easily check that the set of symplectic reflection \mathcal{S} in W , considered as a symplectic reflection group $(\mathfrak{h} \times \mathfrak{h}^*, \omega, W)$, is the same as the set of complex reflections S in W , considered as a complex reflection group (W, \mathfrak{h}) . Therefore, W acts on the symplectic vector space $\mathfrak{h} \times \mathfrak{h}^*$ as a symplectic reflection group if and only if it acts on \mathfrak{h} as a complex reflection group.

The *rational Cherednik algebra*, as introduced by Etingof and Ginzburg [24, page 250], is the symplectic reflection algebra associated to the indecomposable triple $(\mathfrak{h} \times \mathfrak{h}^*, \omega, W)$. In this situation, one can simplify a little the defining relation (1). For each $s \in \mathcal{S}$, fix $\alpha_s \in \mathfrak{h}^*$ to be a basis of the

one dimensional space $\text{Im}(s-1)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathfrak{h}$ a basis of the one dimensional space $\text{Im}(s-1)|_{\mathfrak{h}}$, normalized so that $\alpha_s(\alpha_s^\vee) = 2$. Then the relation (1) can be expressed as:

$$[x_1, x_2] = 0, \quad [y_1, y_2] = 0, \quad [y_1, x_1] = t(y_1, x_1) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)(y_1, \alpha_s)(\alpha_s^\vee, x_1)s, \quad (2)$$

for all $x_1, x_2 \in \mathfrak{h}^*$ and $y_1, y_2 \in \mathfrak{h}$. Notice that these relations imply that $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$ are polynomial subalgebras of $\mathbf{H}_{t,\mathbf{c}}(W)$.

Example 1.15. In the previous example we can take $W = \mathfrak{S}_n$, the symmetric group. Choose a basis x_1, \dots, x_n of \mathfrak{h}^* and dual basis y_1, \dots, y_n of \mathfrak{h} so that

$$\sigma x_i = x_{\sigma(i)}\sigma, \quad \sigma y_i = y_{\sigma(i)}\sigma, \quad \forall \sigma \in \mathfrak{S}_n.$$

Then $\mathcal{S} = \{s_{i,j} \mid 1 \leq i < j \leq n\}$ is the set of all transpositions in \mathfrak{S}_n . This is a single conjugacy class, so $\mathbf{c} \in \mathbb{C}$. Fix

$$\alpha_{i,j} = x_i - x_j, \quad \alpha_{i,j}^\vee = y_i - y_j, \quad \forall 1 \leq i < j \leq n.$$

Then the relations for $\mathbf{H}_{t,\mathbf{c}}(\mathfrak{S}_n)$ become $[x_i, x_j] = [y_i, y_j] = 0$ and

$$\begin{aligned} [y_i, x_j] &= \mathbf{c}s_{i,j}, \quad \forall 1 \leq i < j \leq n, \\ [y_i, x_i] &= t - \mathbf{c} \sum_{j \neq i} s_{i,j}, \quad \forall 1 \leq i \leq n. \end{aligned}$$

Exercise 1.16. To see why the PBW theorem is quite a subtle statement, consider the algebra $\mathbf{L}_{\mathbf{c}}(\mathfrak{S}_2)$ defined to be

$$\mathbb{C}\langle x, y, s \rangle / \langle s^2 = 1, sx = -xs, sy = -ys, [y, x] = 1, (y-s)x = xy + s \rangle.$$

Show that $\mathbf{L}_{\mathbf{c}}(\mathfrak{S}_2) = 0$.

1.4. Double Centralizer property. Let $\mathbf{e} = \frac{1}{|G|} \sum_{g \in G} g$ denote the trivial idempotent in $\mathbb{C}G$. The subalgebra $\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e} \subset \mathbf{H}_{t,\mathbf{c}}(G)$ is called the *spherical subalgebra* of $\mathbf{H}_{t,\mathbf{c}}(G)$. Being a subalgebra, it inherits a filtration from $\mathbf{H}_{t,\mathbf{c}}(G)$. It is a consequence of the PBW theorem that $\text{gr}_{\mathcal{F}}(\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}) \simeq \mathbb{C}[V]^G$. Thus, the spherical subalgebra of $\mathbf{H}_{t,\mathbf{c}}(G)$ is a (not necessarily commutative!) flat deformation of the coordinate ring of V/G - almost exactly what we've been looking for!

The space $\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e}$ is a $(\mathbf{H}_{t,\mathbf{c}}(G), \mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e})$ -bimodule, it is called the Etingof-Ginzburg sheaf. The following result shows that one can recover $\mathbf{H}_{t,\mathbf{c}}(G)$ from knowing $\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}$ and $\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e}$.

Theorem 1.17. (1) The right $\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}$ -module $\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e}$ is reflexive.

$$(2) \text{End}_{\mathbf{H}_{t,\mathbf{c}}(G)}(\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e})^{op} \simeq \mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}.$$

$$(3) \text{End}_{(\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e})^{op}}(\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e}) \simeq \mathbf{H}_{t,\mathbf{c}}(G).$$

Remark 1.18. What is really meant by (1) is that there is an isomorphism of left $\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}$ -modules $\mathbf{eH}_{t,\mathbf{c}}(G) \rightarrow \text{Hom}_{\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}}(\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e}, \mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e})$ and of right $\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}$ -modules $\mathbf{H}_{t,\mathbf{c}}(G)\mathbf{e} \rightarrow \text{Hom}_{\mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e}}(\mathbf{eH}_{t,\mathbf{c}}(G), \mathbf{eH}_{t,\mathbf{c}}(G)\mathbf{e})$.

The above result is extremely useful because, unlike the spherical subalgebra, we have an explicit presentation of $H_{t,\mathbf{c}}(G)$. Therefore, we can try to implicitly study $\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}$ by studying instead the algebra $H_{t,\mathbf{c}}(G)$.

Corollary 1.19. *The algebras $H_{t,\mathbf{c}}(G)$ and $\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}$ are Morita equivalent if and only if $\mathbf{e} \cdot M = 0$ implies $M = 0$ for all $M \in H_{t,\mathbf{c}}(G)\text{-mod}$.*

Proof. Theorem 1.17, together with a basic result in Morita theory e.g. [41, Section 3.5], says that the bimodule $H_{t,\mathbf{c}}(G)\mathbf{e}$ will induce an equivalence of categories $\mathbf{e} \cdot - : H_{t,\mathbf{c}}(G)\text{-mod} \xrightarrow{\sim} \mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}\text{-mod}$ if and only if $H_{t,\mathbf{c}}(G)\mathbf{e}$ is both a generator of the category $H_{t,\mathbf{c}}(G)\text{-mod}$ and a projective $H_{t,\mathbf{c}}(G)\text{-mod}$. Since $H_{t,\mathbf{c}}(G)\mathbf{e}$ is a direct summand of $H_{t,\mathbf{c}}(G)$ it is projective. Therefore we just need to show that it generates the category $H_{t,\mathbf{c}}(G)\text{-mod}$. This condition can be expressed as saying that

$$H_{t,\mathbf{c}}(G)\mathbf{e} \otimes_{\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}} \mathbf{e}M \simeq M, \quad \forall M \in H_{t,\mathbf{c}}(G)\text{-mod}.$$

Equivalently, we require that $H_{t,\mathbf{c}}(G) \cdot \mathbf{e} \cdot H_{t,\mathbf{c}}(G) = H_{t,\mathbf{c}}(G)$. If this is not the case then

$$I := H_{t,\mathbf{c}}(G) \cdot \mathbf{e} \cdot H_{t,\mathbf{c}}(G)$$

is a proper two-sided ideal of $H_{t,\mathbf{c}}(G)$. Hence there exists some module M such that $I \cdot M = 0$. But this is equivalent to $\mathbf{e} \cdot M = 0$. \square

The following notion is very important in the study of rational Cherednik algebras at $t = 1$.

Definition 1.20. The parameter (t, \mathbf{c}) is said to be *aspherical* for G if there exists a non-zero $H_{t,\mathbf{c}}(G)\text{-mod}$ module M such that $\mathbf{e} \cdot M = 0$.

The value $(0, 0)$ is an example of an aspherical value for G .

1.5. The centre of $H_{t,\mathbf{c}}(G)$. One may think of the parameter t as a “quantum parameter”. When $t = 0$, we are in the “quasi-classical situation” and when $t = 1$ we are in the “quantum situation”. The following result gives meaning to such a vague statement. It also shows that the symplectic reflection algebra produces a genuine commutative deformation of the space V/G when $t = 0$.

Theorem 1.21. (1) *If $t = 0$ then the spherical subalgebra $\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}$ is commutative.*

(2) *If $t \neq 0$ then the centre of $\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}$ is \mathbb{C} .*

One can now use the double centralizer property, Theorem 1.17, to lift Theorem 1.21 to a result about the centre of $H_{t,\mathbf{c}}(G)$.

Theorem 1.22 (The Satake isomorphism). *The map $z \mapsto z \cdot \mathbf{e}$ defines an algebra isomorphism $Z(H_{t,\mathbf{c}}(G)) \xrightarrow{\sim} Z(\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e})$ for all parameters (t, \mathbf{c}) .*

Proof. Clearly $z \mapsto z \cdot \mathbf{e}$ is a morphism $Z(H_{t,\mathbf{c}}(G)) \rightarrow Z(\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e})$. Right multiplication on $H_{t,\mathbf{c}}(G) \cdot \mathbf{e}$ by an element a in $Z(\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e})$ defines a right $\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}$ -linear endomorphism of $H_{t,\mathbf{c}}(G) \cdot \mathbf{e}$. Therefore Theorem 1.17 says that there exists some $\zeta(a) \in H_{t,\mathbf{c}}(G)$ such that right multiplication by a equals left multiplication on $H_{t,\mathbf{c}}(G) \cdot \mathbf{e}$ by $\zeta(a)$. The action of a on the right commutes with left multiplication by any element of $H_{t,\mathbf{c}}(G)$ hence $\zeta(a) \in Z(H_{t,\mathbf{c}}(G))$. The homomorphism $\zeta : Z(\mathbf{e}H_{t,\mathbf{c}}(G)\mathbf{e}) \rightarrow Z(H_{t,\mathbf{c}}(G))$ is the inverse to the Satake isomorphism. \square

When $t = 0$, the Satake isomorphism becomes an isomorphism $Z(\mathbf{H}_{0,\mathbf{c}}(G)) \xrightarrow{\sim} \mathbf{e}\mathbf{H}_{0,\mathbf{c}}(G)\mathbf{e}$, and is in fact an isomorphism of Poisson algebras. Theorems 1.21 and 1.22 also imply that $\mathbf{H}_{0,\mathbf{c}}(G)$ is a finite module over its centre. As one might guess, the behavior of symplectic reflection algebras is very different depending on whether $t = 0$ or 1. It is also a very interesting problem to try and relate the representation theory of the algebras $\mathbf{H}_{0,\mathbf{c}}(G)$ and $\mathbf{H}_{1,\mathbf{c}}(G)$ in some meaningful way.

1.6. The Dunkl embedding. In this section, which is designed to be an exercise for the reader, we show how one can use the Dunkl embedding to give easy proofs of many of the important theorems described in the lecture. In particular, one can give elementary proofs of both the PBW theorem and of the fact that the spherical subalgebra is commutative when $t = 0$. Therefore, we let (W, \mathfrak{h}) be a complex reflection group and $\mathbf{H}_{t,\mathbf{c}}(W)$ the associated rational Cherednik algebra. Let $\mathcal{D}_t(\mathfrak{h})$ be the algebra generated by \mathfrak{h} and \mathfrak{h}^* , satisfying the relations

$$[x, x'] = [y, y'] = 0, \quad \forall x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$$

and

$$[y, x] = t(y, x), \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

When $t \neq 0$, $\mathcal{D}_t(\mathfrak{h})$ is isomorphic to $\mathcal{D}(\mathfrak{h})$, the ring of differential operators on \mathfrak{h} . But when $t = 0$, the algebra $\mathcal{D}_t(\mathfrak{h}) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]$ is commutative. Let $\mathfrak{h}_{\text{reg}}$ be the *affine* open subset of \mathfrak{h} on which W acts freely. We can localize $\mathcal{D}_t(\mathfrak{h})$ to $\mathcal{D}_t(\mathfrak{h}_{\text{reg}})$. For each $s \in \mathcal{S}$ define $\lambda_s \in \mathbb{C}^\times$ by $s(\alpha_s) = \lambda_s \alpha_s$. For each $y \in \mathfrak{h}$, the Dunkl operator

$$D_y = y - \sum_{s \in \mathcal{S}} \frac{2\mathbf{c}(s)}{1 - \lambda_s} \frac{(y, \alpha_s)}{\alpha_s} (1 - s)$$

is an element in $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ since α_s is invertible on $\mathfrak{h}_{\text{reg}}$.

Exercise 1.23. (1) Show that the Dunkl operators act on $\mathbb{C}[\mathfrak{h}]$. Hint: it's strongly recommended that you do the example $W = \mathbb{Z}_2$ first, where

$$D_y = y - \frac{\mathbf{c}}{x} (1 - s).$$

(2) Using the fact that

$$s(x) = x - \frac{(\alpha_s^\vee, x)}{2} (1 - \lambda_s) \alpha_s, \quad \forall x \in \mathfrak{h}^*,$$

show that $x \mapsto x$, $w \mapsto w$ and $y \mapsto D_y$ defines a morphism $\mathbf{H}_{t,\mathbf{c}}(W) \rightarrow \mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ i.e. show that the commutation relation

$$[D_y, x] = t(y, x) - \sum_{s \in \mathcal{S}} \mathbf{c}(s) (y, \alpha_s) (\alpha_s^\vee, x) s$$

holds for all $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$. Hint: as above, try the case \mathbb{Z}_2 first.

Now we have everything needed to prove a version PBW theorem for rational Cherednik algebras. The algebra $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ has a natural filtration given by putting $\mathbb{C}[\mathfrak{h}_{\text{reg}}] \rtimes W$ in degree zero and \mathfrak{h} in degree one. Similarly, we define a filtration on the rational Cherednik algebra by putting the generators \mathfrak{h}^* and W in degree zero and \mathfrak{h} in degree one.

Exercise 1.24. Show that the Dunkl embedding is indeed an embedding and that it is compatible with the filtrations on both algebras. Using the fact that the PBW theorem holds for $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$, show that the PBW theorem holds for $H_{t,\mathbf{c}}(W)$. Conclude that $\mathbb{C}[\mathfrak{h}]$ is a faithful $H_{t,\mathbf{c}}(W)$ -module.

The fact that $eH_{t,\mathbf{c}}(G)e$ is commutative when $t = 0$ for an arbitrary symplectic reflection group relies on a very clever but difficult argument by Etingof and Ginzburg. However, for rational Cherednik algebras we have:

Exercise 1.25. By considering its image under the Dunkl embedding, show that the spherical subalgebra $eH_{t,\mathbf{c}}(W)e$ is commutative when $t = 0$.

Recall that a function $f \in \mathbb{C}[\mathfrak{h}]$ is a W -semi-invariant if, for each $w \in W$, $w \cdot f = \alpha_w f$ for some $\alpha_w \in \mathbb{C}$.

Exercise 1.26. Show that the element $\delta := \prod_{s \in \mathcal{S}} \alpha_s \in \mathbb{C}[\mathfrak{h}]$ is a W -semi-invariant. Therefore, there exists some r such that $\delta^r \in \mathbb{C}[\mathfrak{h}]^W$. Show that, after localizing at δ , the Dunkl embedding becomes an isomorphism

$$H_{t,\mathbf{c}}(W)[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W.$$

Exercise 1.27. (Harder) The ring $\mathcal{D}(\mathfrak{h}_{\text{reg}})^W$ is simple. Using this fact, together with the fact that $eH_{1,\mathbf{c}}(W)e$ is an integral domain (why?), show that the centre of $eH_{1,\mathbf{c}}(W)e$ equals \mathbb{C} .

The following fact is also very useful when studying rational Cherednik algebras at $t = 0$.

Exercise 1.28. (1) Assume now that W is a finite Coxeter group. Then there exists a W -equivariant inner product $(-, -)$ on \mathfrak{h} . Show that the rule $x \mapsto \tilde{x} = (x, -)$, $y \mapsto \tilde{y} = (y, -)$ and $w \mapsto w$ defines an automorphism of $H_{t,\mathbf{c}}(W)$, swapping $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$.
(2) Show that $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W$ are central subalgebras of $H_{0,\mathbf{c}}(W)$ (hint: first use the Dunkl embedding to show that $\mathbb{C}[\mathfrak{h}]^W$ is central, then use the automorphism defined in (7)).

1.7. Additional remark.

- In his origin paper, [14], Chevalley showed that if (W, \mathfrak{h}) is a complex reflection group then $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring. The converse was shown by Shephard and Todd in [47]., also books on reflection groups.
- The definition of symplectic reflection algebras first appear in [24].
- The PBW theorem, Theorem 1.13, and its proof are Theorem 1.3 of [24].
- Theorems 1.17 and 1.22 are also contained in [24], as Theorem 1.5 and Theorem 3.1 respectively.
- The first part of Theorem 1.21 is due to Etingof and Ginzburg, [24, Theorem 1.6]. The second part is due to Brown and Gordon, [12, Proposition 7.2]. Both proof rely in a crucial way on the Poisson structure of $\mathbb{C}[V]^G$.

2. RATIONAL CHEREDNIK ALGEBRAS AT $t = 1$

For the remainder of lectures 2 to 4, we will only consider $t = 1$ and omit it from the notation. We will also only be considering rational Cherednik algebras because relatively little is known about general symplectic reflection algebras at $t = 1$. Therefore, we let (W, \mathfrak{h}) be a complex reflection group and $H_{\mathbf{c}}(W)$ the associated rational Cherednik algebra.

As noted in the previous lecture, the centre of $H_{\mathbf{c}}(W)$ equals \mathbb{C} . Therefore, its behavior is very different from the case $t = 0$. If we take $\mathbf{c} = 0$ then $H_0(W) = \mathcal{D}(\mathfrak{h}) \rtimes W$ and the category of modules for $\mathcal{D}(\mathfrak{h}) \rtimes W$ is precisely the category of W -equivariant \mathcal{D} -modules on \mathfrak{h} . In particular, there are *no* finite dimensional representations of this algebra. In general, the algebra $H_{\mathbf{c}}(W)$ has very few finite dimensional representations.

2.1. Recall that the PBW theorem for rational Cherednik algebras implies that, as a vector space, $H_{\mathbf{c}}(W) \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]$. This is an example of a *triangular decomposition*, just like the triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$ encountered in Lie theory, where \mathfrak{g} is a finite dimensional, semi-simple Lie algebra over \mathbb{C} , $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a decomposition into a Cartan subalgebra \mathfrak{h} , nilpotent radical \mathfrak{n}_+ of the Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, and the opposite \mathfrak{n}_- of the nilpotent radical \mathfrak{n}_+ . This suggests that it might be fruitful to try and mimic some of the common constructions in Lie theory. In the representation theory of \mathfrak{g} , one of the categories of modules most intensely studied, and best understood, is category \mathcal{O} , the abelian category generated by all highest weight module. Therefore, it is natural to try and study an analogue of category \mathcal{O} for rational Cherednik algebras. This is what we will do in this lecture.

2.2. **Category \mathcal{O} .** Let $H_{\mathbf{c}}(W)\text{-mod}$ be the category of all finitely generated $H_{\mathbf{c}}(W)$ -modules. It is a hopeless task to try and understand in any detail the whole category $H_{\mathbf{c}}(W)\text{-mod}$. Therefore, one would like to try and understand certain interesting, but manageable, subcategories. The PBW theorem suggests the following very natural definition.

Definition 2.1. *Category \mathcal{O}* is defined to be the full³ subcategory of $H_{\mathbf{c}}(W)\text{-mod}$ consisting of all modules M such that the action of $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ is locally nilpotent.

Remark 2.2. (1) A module M is said to be locally nilpotent for \mathfrak{h} if, for each $m \in M$ there exists some $N \gg 0$ such that $\mathfrak{h}^N \cdot m = 0$.

(2) Each module in category \mathcal{O} is finitely generated as a $\mathbb{C}[\mathfrak{h}]$ -module.

We will give the proofs of the fundamental properties of category \mathcal{O} , since they do not require any sophisticated machinery. However, this does make the lecture rather formal, so we first outline the key features of category \mathcal{O} so that the reader can get their bearings. Recall that an abelian category is called finite length if every object satisfies the ascending chain condition and descending chain condition on subobjects. It is Krull-Schmit if every module has a unique decomposition (up to permuting summands) into a direct sum of indecomposable modules.

³Recall that a subcategory \mathcal{B} of a category \mathcal{A} is called *full* if $\text{Hom}_{\mathcal{B}}(M, N) = \text{Hom}_{\mathcal{A}}(M, N)$ for all $M, N \in \text{Obj}\mathcal{B}$.

- There are only finitely many simple modules in category \mathcal{O} .
- Category \mathcal{O} is a finite length, Krull-Schmit category.
- Every simple module admits a projective cover, hence category \mathcal{O} contains enough projectives.
- Category \mathcal{O} contains “standard modules”, making it a highest weight category.

2.3. Standard objects. One can use induction to construct certain “standard objects” in category \mathcal{O} . The skew-group ring $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ is a subalgebra of $H_{\mathbf{c}}(W)$. Therefore, we can induce to category \mathcal{O} those representations of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ that are locally nilpotent for \mathfrak{h} . Let $\mathfrak{m} = \mathbb{C}[\mathfrak{h}^*]_+$ be the augmentation ideal. Then, for $\lambda \in \text{Irr}(W)$ and $r \in \mathbb{N}$, define the $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ -module $\lambda_r := \mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}^r \otimes \lambda$, where $\mathbb{C}[\mathfrak{h}^*]$ acts only on $\mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}^r$ and W acts diagonally. We set

$$\Delta_r(\lambda) = H_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W} \lambda_r,$$

a module in category \mathcal{O} (check this!). The module $\Delta(\lambda) := \Delta_1(\lambda)$ is called a *standard module* (or, often, a Verma module) of category \mathcal{O} .

Example 2.3. For our favorite example, \mathbb{Z}_2 acting on $\mathfrak{h} = \mathbb{C} \cdot y$ and $\mathfrak{h}^* = \mathbb{C} \cdot x$, we have $\text{Irr}(\mathbb{Z}_2) = \{\rho_0, \rho_1\}$, where ρ_0 is the trivial representation and ρ_1 is the sign representation. Then,

$$\Delta(\rho_0) = \mathbb{C}[x] \otimes \rho_0, \quad \Delta(\rho_1) = \mathbb{C}[x] \otimes \rho_1.$$

The subalgebra $\mathbb{C}[x] \rtimes \mathbb{Z}_2$ acts in the obvious way. The action of y is given as follows

$$y \cdot f(x) \otimes \rho_i = [y, f(x)] \otimes \rho_i + f(x) \otimes y\rho_i = [y, f(x)] \otimes \rho_i, \quad i = 0, 1.$$

Here $[y, f(x)] \in \mathbb{C}[x] \rtimes \mathbb{Z}_2$ is calculated in $H_{\mathbf{c}}(\mathbb{Z}_2)$.

2.4. The Euler element. Let x_1, \dots, x_n be a basis of \mathfrak{h}^* and $y_1, \dots, y_n \in \mathfrak{h}$ the dual basis. Define the Euler element in $H_{\mathbf{c}}(W)$ to be

$$\mathbf{eu} = \sum_{i=1}^n x_i y_i - \sum_{s \in S} \frac{2\mathbf{c}(s)}{1 - \zeta_s} s,$$

where ζ_s is the non-trivial eigenvalue of s acting on \mathfrak{h} i.e. $s \cdot \alpha_s^\vee = \zeta_s \alpha_s^\vee$. The relevance of the element \mathbf{eu} is given by:

Exercise 2.4. Show that $[\mathbf{eu}, x] = x$, $[\mathbf{eu}, y] = -y$ and $[\mathbf{eu}, w] = 0$ for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$.

Therefore \mathbf{eu} defines a \mathbb{Z} -grading on $H_{\mathbf{c}}(W)$, where $\deg(x) = 1$, $\deg(y) = -1$ and $\deg(w) = 0$. The sum $-\sum_{s \in S} \frac{2\mathbf{c}_s}{1 - \zeta_s} s$ belongs to $Z(W)$, the centre of the group algebra. Therefore, if λ is an irreducible W -module, this central element will act by a scalar on λ . This scalar will be denoted \mathbf{c}_λ .

Lemma 2.5. *Each $M \in \mathcal{O}$ is the direct sum of its generalized \mathbf{eu} -eigenspaces*

$$M = \bigoplus_{a \in \mathbb{C}} M_a,$$

and $\dim M_a < \infty$ for all $a \in \mathbb{C}$.

Proof. Since M is in category \mathcal{O} , we can choose some finite dimensional $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ -submodule M' of M that generates M as a $H_c(W)$ -module. Since M' is finite dimensional, there exists some $r \gg 0$ such that $\mathfrak{h}^r \cdot M' = 0$. Thus we may find $\lambda_1, \dots, \lambda_k \in \text{Irr}(W)$ such that the sequence

$$\bigoplus_{i=1}^k \Delta_r(\lambda_i) \rightarrow M \rightarrow 0$$

is exact. One can check directly that each $\Delta_r(\lambda_i)$ is a direct sum of its generalized **eu**-eigenspaces, with each eigenspace finite dimensional. This implies that M has this property too. \square

Exercise 2.6. (1) Give an example of a module $M \in H_c(W)\text{-mod}$ that is not the direct sum of its generalized **eu**-eigenspaces.

(2) Using **eu**, show that every finite dimensional $H_c(W)$ -module is in category \mathcal{O} (hint: how does $y \in \mathfrak{h}$ act on a generalized eigenspace for **eu**?).

2.5. Characters. Using the Euler operator **eu** we can define the character of a module $M \in \mathcal{O}$ by

$$\text{ch}(M) = \sum_{a \in \mathbb{C}} (\dim M_a) t^a.$$

The Euler element acts via the scalar \mathbf{c}_λ on $1 \otimes \lambda \subset \Delta(\lambda)$. This implies that

$$\text{ch}(\Delta(\lambda)) = \frac{\dim(\lambda) t^{\mathbf{c}_\lambda}}{(1-t)^n}.$$

Exercise 2.7. Show that $\text{ch}(M) \in \bigoplus_{a \in \mathbb{C}} t^a \mathbb{Z}[[t]]$ for all $M \in \mathcal{O}$. (Harder) As shown in exercise 2.20 below, the standard modules $\Delta(\lambda)$ are a \mathbb{Z} -basis of the Grothendieck group $K_0(\mathcal{O})$. Using this fact, show that

$$\text{ch}(M) = \frac{1}{(1-t)^n} \cdot f(t)$$

where $f(t) \in \mathbb{Z}[x^a \mid a \in \mathbb{C}]$, the group ring of the additive group $(\mathbb{C}, +)$.

The above exercise shows that $M \mapsto (1-t)^n \cdot \text{ch}(M)$ is a morphism of abelian groups $K_0(\mathcal{O}) \rightarrow \mathbb{Z}[(\mathbb{C}, +)]$. It is not in general an embedding.

2.6. Simple modules. The basic problems motivating research in the theory of rational Cherednik algebras are:

- (1) Classify the simple modules in \mathcal{O} .
- (2) Calculate $\text{ch}(L)$ for all simple modules $L \in \mathcal{O}$.

The first problem is easy, but the second is very difficult (and still open in general).

Lemma 2.8. *Let M be a non-zero module in category \mathcal{O} . Then, there exists some $\lambda \in \text{Irr}(W)$ and non-zero homomorphism $\Delta(\lambda) \rightarrow M$.*

Proof. Note that the real part of the weights of M are bounded from below i.e. there exists some $K \in \mathbb{R}$ such that $M_a \neq 0$ implies $\text{Re}(a) \geq K$. Therefore we may choose some $a \in \mathbb{C}$ such that $M_a \neq 0$ and $M_b = 0$ for all $b \in \mathbb{C}$ such that $a - b \in \mathbb{R}_{>0}$. An element $m \in M$ is said to be *singular* if $\mathfrak{h} \cdot m = 0$ i.e. it is annihilated by all y 's. Our assumption implies that all elements in M_a are

singular. If λ occurs in M_a with non-zero multiplicity then there is a well defined homomorphism $\Delta(\lambda) \rightarrow M$, whose restriction to $1 \otimes \lambda$ injects into M_a . \square

Lemma 2.9. *Each standard module $\Delta(\lambda)$ has a simple head $L(\lambda)$ and the set*

$$\{L(\lambda) \mid \lambda \in \text{Irr}(W)\}$$

is a complete set of non-isomorphic simple modules of category \mathcal{O} .

Proof. Let R be the sum of all proper submodules of $\Delta(\lambda)$. It suffices to show that $R \neq \Delta(\lambda)$. The weight subspace $\Delta(\lambda)_{\mathbf{c}_\lambda} = 1 \otimes \lambda$ is irreducible as a W -module and generates $\Delta(\lambda)$. If $R_{\mathbf{c}_\lambda} \neq 0$ then there exists some proper submodule N of $\Delta(\lambda)$ such that $N_{\mathbf{c}_\lambda} \neq 0$. But then $N = \Delta(\lambda)$. Thus, $R_{\mathbf{c}_\lambda} = 0$, implying that R is a proper submodule of $\Delta(\lambda)$. Now let L be a simple module in category \mathcal{O} . By Lemma 2.8, there exists a non-zero homomorphism $\Delta(\lambda) \rightarrow L$ for some $\lambda \in \text{Irr}(W)$. Hence $L \simeq L(\lambda)$. The fact that $L(\lambda) \simeq L(\mu)$ implies $\lambda \simeq \mu$ follows from the fact that L_{sing} , the space of singular vectors in L , is irreducible as a W -module. \square

Example 2.10. Let's consider $H_{\mathbf{c}}(\mathbb{Z}_2)$ at $\mathbf{c} = \frac{-3}{2}$. Then, one can check that

$$\Delta(\rho_1) = \mathbb{C}[x] \otimes \rho_1 \twoheadrightarrow L(\rho_1) = (\mathbb{C}[x] \otimes \rho_1) / (x^3 \mathbb{C}[x] \otimes \rho_1).$$

On the other hand, $\Delta(\rho_0) = L(\rho_0)$ is simple. The composition series of $\Delta(\rho_1)$ is $\begin{smallmatrix} L(\rho_1) \\ L(\rho_0) \end{smallmatrix}$.

Corollary 2.11. *Every module in category \mathcal{O} has finite length.*

Proof. Let M be a non-zero object of category \mathcal{O} . Choose some real number $K \gg 0$ such that $\text{Re}(\mathbf{c}_\lambda) < K$ for all $\lambda \in \text{Irr}(W)$. We write $M^{\leq K}$ for the sum of all weight spaces M_a such that $\text{Re}(a) \leq K$. It is a finite dimensional subspace. Lemma 2.8 implies that $N^{\leq K} \neq 0$ for all non-zero submodules N of M . Therefore, if $N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots$ is a proper descending chain of submodule of M then $N_0^{\leq K} \supsetneq N_1^{\leq K} \supsetneq N_2^{\leq K} \supsetneq \cdots$ is a proper descending chain of subspaces of $M^{\leq K}$. Hence the chain must have finite length. \square

2.7. Projective modules. A module $P \in \mathcal{O}$ is said to be projective if the functor $\text{Hom}_{H_{\mathbf{c}}(W)}(P, -) : \mathcal{O} \rightarrow \text{Vect}(\mathbb{C})$ is exact.

Definition 2.12. An object Q in \mathcal{O} is said to have a Δ -filtration if it has a finite filtration $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q$ such that $Q_i/Q_{i-1} \simeq \Delta(\lambda_i)$ for some $\lambda_i \in \text{Irr}(W)$ and all $1 \leq i \leq r$.

The following theorem, first shown in [33], is of key importance in the study of category \mathcal{O} . We follow the proof given in [1].

Theorem 2.13. *Every simple module $L(\lambda)$ in category \mathcal{O} has a projective cover $P(\lambda)$. Moreover, each $P(\lambda)$ has a finite Δ -filtration.*

For $a \in \mathbb{C}$ we denote by \bar{a} its image in \mathbb{C}/\mathbb{Z} . We write $\mathcal{O}^{\bar{a}}$ for the full subcategory of \mathcal{O} consisting of all M such that $M_b = 0$ for all $b \notin a + \mathbb{Z}$.

Exercise 2.14. Using the fact that all weights of $H_c(W)$ for the adjoint action of \mathbf{eu} are in \mathbb{Z} , show that

$$\mathcal{O} = \bigoplus_{\bar{a} \in \mathbb{C}/\mathbb{Z}} \mathcal{O}^{\bar{a}}.$$

Proof of Theorem 2.13. The above exercise shows that it suffices to construct a projective cover $P(\lambda)$ for $L(\lambda)$ in $\mathcal{O}^{\bar{a}}$. Fix a representative $a \in \mathbb{C}$ of \bar{a} . For each $k \in \mathbb{Z}$, let $\mathcal{O}^{\geq k}$ denote the full subcategory of $\mathcal{O}^{\bar{a}}$ consisting of modules M such that $M_b \neq 0$ implies that $b - a \in \mathbb{Z}_{\geq k}$. Then, for $k \gg 0$, we have $\mathcal{O}^{\geq k} = 0$ and for $k \ll 0$ we have $\mathcal{O}^{\geq k} = \mathcal{O}^{\bar{a}}$. Our proof will be by induction on k . Namely, for each k and all $\lambda \in \text{Irr}(W)$ such that $L(\lambda) \in \mathcal{O}^{\geq k}$, we will construct a projective cover $P_k(\lambda)$ of $L(\lambda)$ in $\mathcal{O}^{\geq k}$ such that $P_k(\lambda)$ is a *quotient* of an object of \mathcal{O} equipped with a Δ -filtration. Let k_0 be the largest integer such that $\mathcal{O}^{\geq k_0} \neq 0$.

Exercise 2.15. Show that $\mathcal{O}^{\geq k_0}$ is semi-simple with $P_{k_0}(\lambda) = \Delta(\lambda) = L(\lambda)$ for all λ such that $L(\lambda) \in \mathcal{O}^{\geq k_0}$.

We assume that we have constructed, for all $L(\lambda) \in \mathcal{O}^{\geq k+1}$, a projective cover $P_{k+1}(\lambda)$ of $L(\lambda)$ in $\mathcal{O}^{\geq k+1}$ with the desired properties. If $\mathcal{O}^{\geq k+1} = \mathcal{O}^{\geq k}$ then there is nothing to do so we may assume that there exist $\mu_1, \dots, \mu_r \in \text{Irr}(W)$ such that $L(\mu_i) \in \mathcal{O}^{\geq k} \setminus \mathcal{O}^{\geq k+1}$. Note that $\mathbf{c}_{\mu_i} = a + k$ for all i . For all $M \in \mathcal{O}^{\geq k}$, either $M_{a+k} = 0$ (in which case $M \in \mathcal{O}^{\geq k+1}$) or M_{a+k} consists of singular vectors i.e. $\mathfrak{h} \cdot M_{a+k} = 0$. Therefore we have $P_k(\mu_i) = \Delta(\mu_i)$ for $1 \leq i \leq r$. Thus, we are left with constructing $P_k(\lambda)$ for all those λ such that $L(\lambda) \in \mathcal{O}^{\geq k+1}$.

Claim 2.16. There exists some integer $N \gg 0$ such that $(\mathbf{eu} - \mathbf{c}_\lambda)^N \cdot m = 0$ for all $M \in \mathcal{O}^{\geq k}$ and all $m \in M_{\mathbf{c}_\lambda}$.

Proof of the claim. Since $\dim(M_{a+k}) < \infty$ there exist $n_i \in \mathbb{N}$ and a morphism

$$\phi : \bigoplus_{i=1}^r \Delta(\mu_i)^{\oplus n_i} \longrightarrow M$$

such that the cokernel M' of ϕ is in $\mathcal{O}^{\geq k+1}$. Therefore we may construct a surjection

$$\psi : \bigoplus_{\eta} P_{k+1}(\eta)^{s_\eta} \twoheadrightarrow M'$$

where the sum is over all $\eta \in \text{Irr}(W)$ such that $L(\eta) \in \mathcal{O}^{\geq k+1}$. The induction hypothesis guarantees that each $Q_{k+1}(\eta)$ has a Δ -filtration. This implies that there is some $N \gg 0$ such that $(\mathbf{eu} - \mathbf{c}_\lambda)^{N-1} \cdot q = 0$ for all $q \in Q_{k+1}(\eta)_{\mathbf{c}_\lambda}$. Therefore $(\mathbf{eu} - \mathbf{c}_\lambda)^{N-1} \cdot m$ lies in the image of ϕ for all $m \in M_{\mathbf{c}_\lambda}$ and hence $(\mathbf{eu} - \mathbf{c}_\lambda)^N \cdot m = 0$. \square

Now choose some integer $r \gg (a + k) - \mathbf{c}_\lambda$ and define

$$R(\lambda) = \frac{\Delta_r(\lambda)}{H_c(W) \cdot (\mathbf{eu} - \mathbf{c}_\lambda)^N (1 \otimes 1 \otimes \lambda)}.$$

Then $\text{Hom}_{H_c(W)}(R(\lambda), -)$ is exact on $\mathcal{O}^{\geq k}$. It is non-zero because it surjects onto $\Delta(\lambda)$. The only problem is that it does not belong to $\mathcal{O}^{\geq k}$. So we let $\tilde{R}(\lambda)$ be the $H_c(W)$ -submodule generated by all weight spaces $R(\lambda)_b$ with $b - a \notin \mathbb{Z}_{\geq k}$. Then $P_k(\lambda) := R(\lambda)/\tilde{R}(\lambda)$. By construction, it belongs

to $\mathcal{O}^{\geq k}$ and if $f : R(\lambda) \rightarrow M$ is any morphism with $M \in \mathcal{O}^{\geq k}$ then $\tilde{R}(\lambda) \subset \text{Ker } f$. Therefore it is the projective cover of $L(\lambda)$ in $\mathcal{O}^{\geq k}$. We have constructed $P_k(\lambda)$ as a quotient of $\Delta_r(\lambda)$, an object equipped with a Δ -filtration.

The only thing left to show is that if $k \ll 0$ such that $\mathcal{O}^{\geq k} = \mathcal{O}^{\bar{a}}$ then $P_k(\lambda)$ has a Δ -filtration. By construction, it is a quotient of an object $M \in \mathcal{O}$ that is equipped with a Δ -filtration. But our assumption on k means that $P_k(\lambda)$ is projective in \mathcal{O} . Thus, it is a direct summand of M . Therefore, it suffices to note that if $M = M_1 \oplus M_2$ is an object of \mathcal{O} equipped with a Δ -filtration, then each M_i also has a Δ -filtration (this follows from the fact that the modules $\Delta(\lambda)$ are *indecomposable*). \square

In general, it is very difficult to explicitly construct the projective covers $P(\lambda)$. The object $P = \bigoplus_{\lambda \in \text{Irr}(W)} P(\lambda)$ is a projective generator of \mathcal{O} . Therefore, we have an equivalence of abelian categories

$$\mathcal{O} \simeq A\text{-mod},$$

where $A = \text{End}_{\text{Hc}(W)}(P)$ is a finite dimensional algebra.

2.8. Highest weight categories. Just as for category \mathcal{O} of a semi-simple Lie algebra \mathfrak{g} over \mathbb{C} , the existence of standard modules in category \mathcal{O} implies that this category has a lot of additional structure. In particular, it is an example of a highest weight (or quasi-hereditary) category. The abstract notion of a highest weight category was introduced in [15].

Definition 2.17. Let \mathcal{A} be an abelian, \mathbb{C} -linear and finite length category, and Λ a poset. We say that (\mathcal{A}, Λ) is a *highest weight category* if

- (1) There is a complete set $\{L(\lambda) \mid \lambda \in \Lambda\}$ of non-isomorphic simple objects labeled by Λ .
- (2) There is a collection of *standard* objects $\{\Delta(\lambda) \mid \lambda \in \Lambda\}$ of \mathcal{A} , with surjections $\phi_\lambda : \Delta(\lambda) \rightarrow L(\lambda)$ such that all composition factors $L(\mu)$ of $\text{Ker } \phi_\lambda$ satisfy $\mu < \lambda$.
- (3) Each $L(\lambda)$ has a projective cover $P(\lambda)$ in \mathcal{A} and the projective cover $P(\lambda)$ admits a Δ -filtration $0 = F_0 P(\lambda) \subset F_1 P(\lambda) \subset \cdots \subset F_m P(\lambda) = P(\lambda)$ such that
 - $F_m P(\lambda) / F_{m-1} P(\lambda) \simeq \Delta(\lambda)$.
 - For $0 < i < m$, $F_i P(\lambda) / F_{i-1} P(\lambda) \simeq \Delta(\mu)$ for some $\mu > \lambda$.

Define a partial ordering on $\text{Irr}(W)$ by setting

$$\lambda \leq_{\mathbf{c}} \mu \iff \mathbf{c}_\mu - \mathbf{c}_\lambda \in \mathbb{Z}_{\geq 0}.$$

Lemma 2.18. *Let $\lambda, \mu \in \text{Irr}(W)$ such that $\lambda \not\leq_{\mathbf{c}} \mu$. Then $\text{Ext}_{\text{Hc}(W)}^1(\Delta(\lambda), \Delta(\mu)) = 0$.*

Proof. Assume that we are given a short exact sequence

$$0 \rightarrow \Delta(\mu) \rightarrow M \rightarrow \Delta(\lambda) \rightarrow 0$$

for some $M \in \text{Hc}(W)\text{-mod}$. Then $M \in \mathcal{O}$. Take $0 \neq v \in \Delta(\lambda)_{\mathbf{c}_\lambda} = 1 \otimes \lambda$ and $m \in M_{\mathbf{c}_\lambda}$ that maps onto v . Since $\lambda \not\leq_{\mathbf{c}} \mu$, there is no $a \in \mathbb{C}$ such that $a - \mathbf{c}_\lambda \in \mathbb{Z}_{>0}$ and $M_a \neq 0$. Hence $\mathfrak{h} \cdot m = 0$. This implies that the above sequence splits. \square

Theorem 2.19. *Category \mathcal{O} is a highest weight category under the ordering $\leq_{\mathbf{c}}$.*

Proof. The only thing left to check is that we can choose a Δ -filtration on the projective covers $P(\lambda)$ such that the conditions of Definition 2.17 (3) are satisfied. By Theorem 2.13, we can always choose some Δ -filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_m = P(\lambda)0$ with $F_i/F_{i-1} \simeq \Delta(\mu_i)$. Since $P(\lambda)$ surjects onto $L(\lambda)$, we must have $\mu_m = \lambda$. I claim that one can always choose the μ_i such that $\mu_i \geq_{\mathbf{c}} \mu_{i+1}$ for all $0 < i < m$. The proof is by induction on i . But first we remark that the fact that $P(\lambda)$ is indecomposable implies that all μ_i are comparable under $<_{\mathbf{c}}$. Assume that $\mu_j \geq_{\mathbf{c}} \mu_{j+1}$ for all $j < i$. If $\mu_i <_{\mathbf{c}} \mu_{i+1}$ then it suffices to show that there is another Δ -filtration of $P(\lambda)$ with composition factors μ'_j such that $\mu'_j = \mu_j$ for all $j \neq i, i+1$ and $\mu'_i = \mu_{i+1}$, $\mu'_{i+1} = \mu_i$. We have

$$0 \rightarrow F_i \rightarrow F_{i+1} \rightarrow \Delta(\mu_{i+1}) \rightarrow 0$$

which quotienting out by F_{i-1} gives

$$0 \rightarrow \Delta(\mu_i) \rightarrow F_{i+1}/F_{i-1} \rightarrow \Delta(\mu_{i+1}) \rightarrow 0.$$

Lemma 2.18 implies that the above sequence splits. Hence $F_{i+1}/F_{i-1} \simeq \Delta(\mu_i) \oplus \Delta(\mu_{i+1})$. Thus, we may choose $F_{i-1} \subset F'_i \subset F_i$ such that $F'_i/F_{i-1} \simeq \Delta(\mu_{i+1})$ and $F_{i+1}/F'_i \simeq \Delta(\mu_i)$ as required. Hence the claim is proved. This means that $\mu_{m-1} \geq_{\mathbf{c}} \lambda$. If $\mu_{m-1} =_{\mathbf{c}} \lambda$ then Lemma 2.18 implies that $P(\lambda)/F_{m-2} \simeq \Delta(\mu_{m-1}) \oplus \Delta(\lambda)$ and hence the head of $P(\lambda)$ is not simple. This contradicts the fact that $P(\lambda)$ is a projective cover. \square

Exercise 2.20. Using the fact that \mathcal{O} is a highest weight category, show that the standard modules $\Delta(\lambda)$ are a \mathbb{Z} -basis of the Grothendieck group $K_0(\mathcal{O})$.

A corollary of Theorem 2.19 is that Bernstein-Gelfand-Gelfand (BGG) reciprocity holds in category \mathcal{O} .

Corollary 2.21 (BGG-reciprocity). *For $\lambda, \mu \in \text{Irr} W$,*

$$(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)].$$

Corollary 2.22. *The global dimension of \mathcal{O} is finite.*

Exercise 2.23. (Harder) Prove Corollary 2.22 (hint: first prove by induction that $\text{p.d.}(\Delta(\lambda)) < \infty$ for all λ , then apply a similar inductive argument to show that $\text{p.d.}(L(\lambda)) < \infty$).

Corollary 2.24. *Category \mathcal{O} is semi-simple if and only if $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \text{Irr}(W)$.*

Proof. Since $\Delta(\lambda)$ is indecomposable and $L(\lambda)$ a quotient of $\Delta(\lambda)$, it is clear that \mathcal{O} semi-simple implies that $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \text{Irr}(W)$. Conversely, BGG reciprocity implies that $P(\lambda) = L(\lambda)$ for all λ which implies that \mathcal{O} is semi-simple. \square

Exercise 2.25. Show that if $\mathbf{c}_\lambda - \mathbf{c}_\mu \notin \mathbb{Z}_{>0}$ for all $\lambda \neq \mu \in \text{Irr}(W)$ then category \mathcal{O} is semi-simple. Conclude that \mathcal{O} is semi-simple for generic parameters \mathbf{c} .

2.9. Category \mathcal{O} for \mathbb{Z}_2 . The idea of this section, which consists mainly of exercises, is simply to try and better understand category \mathcal{O} when $W = \mathbb{Z}_2$. Recall that, in this case, $W = \langle s \rangle$ with $s^2 = 1$ and the defining relations for $\mathbf{H}_{\mathbf{c}}(\mathbb{Z}_2)$ are $sx = -xs$, $sy = -ys$ and

$$[y, x] = 1 - 2\mathbf{c}s.$$

- Exercise 2.26.* (1) For each \mathbf{c} , describe the simple modules $L(\lambda)$ as quotients of $\Delta(\lambda)$. For which values of \mathbf{c} is category \mathcal{O} semi-simple?
- (2) For each \mathbf{c} , describe the partial ordering on $\text{Irr}(W)$ coming from the highest weight structure on \mathcal{O} .

In this case, one can also describe explicitly what the projective covers of the simple modules are, though this is a tricky calculation.

Exercise 2.27. (Harder) Calculate the projective covers $P(\lambda)$ (hint: Use BGG reciprocity to calculate the rank of $P(\lambda)$ as a (free) $\mathbb{C}[x]$ -module).

Recall that a finite dimensional \mathbb{C} -algebra A is said to be *basic* if the dimension of all simple A -modules is one. Every basic algebra can be described as a quiver with relations. One way to reconstruct A from $A\text{-mod}$ is via the isomorphism

$$A = \text{End}_A \left(\bigoplus_{\lambda \in \text{Irr}(A)} P(\lambda) \right),$$

where $\text{Irr}(A)$ is the set of isomorphism classes of simple A -modules and $P(\lambda)$ is the projective cover of λ .

Exercise 2.28. Construct a basic A in terms of a quiver with relations such that $A\text{-mod} \simeq \mathcal{O}$. Hint: use BGG-reciprocity to calculate the dimension of A .

Assume now that W is any complex reflection group. Recall that \mathbf{c} is said to be *aspherical* if there exists some non-zero $M \in \mathbf{H}_{\mathbf{c}}(W)\text{-mod}$ such that $\mathbf{e} \cdot M = 0$. As in Lie theory, we have a “Generalized Duflo Theorem”:

Theorem 2.29. *Let J be a primitive ideal in $\mathbf{H}_{\mathbf{c}}(W)$. Then there exists some $\lambda \in \text{Irr}(W)$ such that*

$$J = \text{Ann}_{\mathbf{H}_{\mathbf{c}}(W)}(L(\lambda)).$$

- Exercise 2.30.* (1) Using the Generalized Duflo Theorem, and arguments as in the proof of Corollary 1.17 of lecture one, show that \mathbf{c} is aspherical if and only if there exists a simple module $L(\lambda)$ in category \mathcal{O} such that $\mathbf{e} \cdot L(\lambda) = 0$.
- (2) Calculate the aspherical values for $W = \mathbb{Z}_2$.

2.10. Additional remark.

- The results of this lecture all come from (at least) one of the papers [21], [33] or [27].
- Theorem 2.19 is shown in [27].
- The fact that BGG reciprocity, Corollary 2.21, follows from Theorem 2.19 is shown in [27, Proposition 3.3].
- The definition given in [15, Definition 3.1] is dual to the one give in Definition 2.17. It is also given in much greater generality.
- The generalized Duflo theorem, Theorem 2.29, is given in [26].

3. THE SYMMETRIC GROUP

In this lecture we concentrate on category \mathcal{O} for $W = \mathfrak{S}_n$, the symmetric group. The reason for this is that category \mathcal{O} is much better understood for this group than for other complex reflection groups, though there are still several open problems. For instance, it is known for which parameters \mathbf{c} the algebra $H_{\mathbf{c}}(\mathfrak{S}_n)$ admits finite dimensional representations, and for any such parameter, exactly how many non-isomorphic simple modules there are, see [6] - it turns out that $H_{\mathbf{c}}(\mathfrak{S}_n)$ admits at most one finite dimensional, simple module. In fact, we now have a good understanding, [55], of the “size” (i.e. Gelfand-Kirillov dimension) of all simple modules in category \mathcal{O} .

But, in this lecture, we will concentrate on the main problem mentioned in lecture two, that of calculating the multiplicities of simple modules in composition series for standard modules. For the symmetric group, we now also have a complete answer to this question: the multiplicities are given by evaluating at one the transition matrices between standard and canonical basis of a certain Fock space. The proof of this remarkable fact involves the work of several people, namely Rouquier, Varagnolo-Vasserot and Leclerc-Thibon. It relies upon making links between category \mathcal{O} , ν -Schur algebras and quantum affine algebras. As a consequence, it is possible, in theory, to calculate the character of the simple modules $L(\lambda)$ in category \mathcal{O} .

3.1. The rational Cherednik algebra associated to the symmetric group. Recall from example 1.15 that the rational Cherednik algebra associated to the symmetric group \mathfrak{S}_n is the quotient of

$$T(\mathbb{C}^{2n}) \rtimes \mathfrak{S}_n = \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes \mathfrak{S}_n$$

by the relations

$$\begin{aligned} [x_i, x_j] &= 0, \quad [y_i, y_j] = 0, \quad \forall i, j, \\ [y_i, x_j] &= \mathbf{c} s_{ij}, \quad \forall i \neq j, \end{aligned}$$

and

$$[y_i, x_i] = 1 - \mathbf{c} \sum_{j \neq i} s_{ij}.$$

In this case, the Euler element is⁴

$$\mathbf{eu} = \frac{1}{2} \sum_{i=1}^n x_i y_i + y_i x_i = \sum_{i=1}^n x_i y_i + \frac{n}{2} - \frac{\mathbf{c}}{2} \sum_{1 \leq i \neq j \leq n} s_{i,j}.$$

3.2. Representations of \mathfrak{S}_n . It is a classical result, going back to Schur, that the irreducible representations of the symmetric group over \mathbb{C} are naturally labeled by partitions of n . Therefore, we can (and will) identify $\text{Irr}(\mathfrak{S}_n)$ with \mathcal{P}_n , the set of all partitions of n and denote by λ both a partition of n and the corresponding representation of \mathfrak{S}_n . For more on the construction of the representations of \mathfrak{S}_n , see [25].

Example 3.1. The partition (n) labels the trivial representation and (1^n) labels the sign representation. The reflection representation \mathfrak{h} is labeled by $(n-1, 1)$. More generally, each of the

⁴The Euler element defined here differs from the one in section 2.4 by a constant.

representations $\bigwedge^i \mathfrak{h}$ is an irreducible \mathfrak{S}_n -module and is labeled by $(n - i, 1^i)$. Note that the trivial representation is $\bigwedge^0 \mathfrak{h}$ and the sign representation is just $\bigwedge^{n-1} \mathfrak{h}$.

Associated to partitions is a wealth of beautiful combinatorics. We'll need to borrow a little of this combinatorics. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. Associated to λ is the *partition statistic*, which is defined to be

$$n(\lambda) := \sum_{i=1}^k (i-1)\lambda_i.$$

We visualize λ as a certain array of boxes, called a *Young tableau*, as in the example⁵ $\lambda = (4, 3, 1)$:

1			
2	0	1	
0	1	2	0

To be precise, the Young diagram of λ is $Y(\lambda) := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq k, 1 \leq i \leq \lambda_j\} \subset \mathbb{Z}^2$. The *content* of the box (i, j) is $\text{cont}(i, j) := i - j$. A *removable* box is a box of the boundary of λ which can be removed, leaving a partition of $|\lambda| - 1$. An *indent* box is a concave corner on the rim of λ where a box can be added, giving a partition of $|\lambda| + 1$. For instance, $\lambda = (4, 3, 1)$ has three removable boxes (with content 2, -1 and -3), and four indent boxes (with content 3, 1, -2 and -4).

There is a natural partial ordering on \mathcal{P}_n , the set of all partitions of n , which is the *dominance ordering* and is defined by $\lambda \leq \mu$ if and only if

$$\lambda_1 + \dots + \lambda_k \leq \mu_1 + \dots + \mu_k, \quad \forall k.$$

Finally, we require the notion of r -rim-hooks and r -cores. An r -rim-hook of λ is a *connected* skew partition $\lambda \setminus \mu$ of length r that does not contain the subpartition $(2, 2)$ i.e. it is a segment of length r of the edge of λ . For example, $\{(2, 2), (3, 2), (3, 1), (4, 1)\}$ is a 4-rim-hook of $(4, 3, 1)$. The r -core of λ is the partition $\mu \subset \lambda$ obtained by removing, one after another, all possible r -rim-hooks of λ . It is known that the r -core is independent of the order in which the hooks are removed. For example, the 3-core of $(4, 3, 1)$ is (2) .

Exercise 3.2. Write a program that calculates the r -core of a partition.

3.3. Recall that the Euler element \mathbf{eu} acts on the space $1 \otimes \lambda \subset \Delta(\lambda)$ by a scalar, denoted \mathbf{c}_λ .

Lemma 3.3. *For each $\lambda \vdash n$ and $\mathbf{c} \in \mathbb{C}$,*

$$\mathbf{c}_\lambda = \frac{n}{2} + \mathbf{c}(n(\lambda) - n(\lambda')). \quad (3)$$

Proof. The *Jucys-Murphy* elements in $\mathbb{C}[\mathfrak{S}_n]$ are defined to be $\Theta_i = \sum_{j < i} s_{ij}$, for all $i = 2, \dots, n$ so that

$$\mathbf{eu} = \sum_{i=1}^n x_i y_i + \frac{n}{2} - \mathbf{c} \sum_{i=2}^n \Theta_i.$$

⁵The numbers in the boxes are the residues of λ modulo 3, see section 3.7.

Let σ be a standard tableau of shape λ and v_σ the corresponding vector in χ_λ . Then

$$\Theta_i \cdot v_\sigma = \text{ct}_\sigma(i) v_\sigma,$$

where $c_\sigma(i)$ is the column of λ containing i , $r_\sigma(i)$ is the row of λ containing i and $\text{ct}_\sigma(i) := c_\sigma(i) - r_\sigma(i)$ is the content of the node containing i . Note that $\text{ct}_\sigma(1) = 0$ for all standard tableaux σ . Therefore

$$\mathbf{eu} \cdot v_\sigma = \left(\frac{n}{2} - \mathbf{c} \sum_{i=2}^n \text{ct}_\sigma(i) \right) v_\sigma,$$

and hence

$$\mathbf{c}_\lambda = \frac{n}{2} - \mathbf{c} \sum_{i=2}^n \text{ct}_\sigma(i) = \frac{n}{2} - \mathbf{c} \sum_{i=1}^n \text{ct}_\sigma(i).$$

Now $\sum_{i=1}^n r_\sigma(i) = \sum_{j=1}^{\ell(\lambda)} (j-1)\lambda_j = n(\lambda)$ and similarly $\sum_{i=1}^n c_\sigma(i) = n(\lambda')$. This implies equation (3). \square

3.4. The ν -Schur algebra. Recall that the category \mathcal{C}_n of finite-dimensional representations of \mathfrak{gl}_n , or equivalently of its enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$, is semi-simple. The simple modules in this category are L_λ where $\lambda \in \mathbb{Z}^n$ such that $\lambda_i - \lambda_{i+1} \geq 0$ for all $1 \leq i \leq n-1$. The set $\mathcal{P}(n)$ of all partitions with length at most n is a subset of this set. Let V denote the vectorial representation of \mathfrak{gl}_n . For each $d \geq 1$ there is an action of \mathfrak{gl}_n on $V^{\otimes d}$. The symmetric group also acts on $V^{\otimes d}$ on the right by

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}, \quad \forall \sigma \in \mathfrak{S}_d.$$

It is known that these two actions commute. Thus, we have homomorphisms $\phi_d : \mathcal{U}(\mathfrak{gl}_n) \rightarrow \text{End}_{\mathbb{C}\mathfrak{S}_d}(V^{\otimes d})$ and $\psi_d : \mathbb{C}\mathfrak{S}_d \rightarrow \text{End}_{\mathcal{U}(\mathfrak{gl}_n)}(V^{\otimes d})^{op}$. Schur-Weyl duality says that

Proposition 3.4. *The homomorphisms ϕ_d and ψ_d are surjective for all $d \geq 1$.*

Let $\mathbf{S}(n, d)$ be the image of ϕ_d . It is called the Schur algebra. We denote by $\mathcal{C}_n(d)$ the full subcategory of \mathcal{C}_n consisting of all modules whose composition factors are of the form L_λ for $\lambda \in \mathcal{P}_d(n)$, where $\mathcal{P}_d(n)$ is the set of all partitions of d that belong to $\mathcal{P}(n)$. It is easy to check that $[V^{\otimes d} : L_\lambda] \neq 0$ if and only if $\lambda \in \mathcal{P}_d(n)$. Moreover, it is known that $\mathcal{C}_n(d) \simeq \mathbf{S}(n, d)\text{-mod}$.

The above construction can be quantized. Let $\nu \in \mathbb{C}^\times$. Then, the *quantized enveloping algebra* $\mathcal{U}_\nu(\mathfrak{gl}_n)$ is a deformation of $\mathcal{U}(\mathfrak{gl}_n)$. The quantum enveloping algebra $\mathcal{U}_\nu(\mathfrak{gl}_n)$ still acts on V . The group algebra $\mathbb{C}\mathfrak{S}_n$ also has a natural deformation, the *Hecke algebra* of type A , denoted $\mathcal{H}_\nu(d)$. This algebra is described in example 4.3. As one might expect, it is also possible to deform the action of $\mathbb{C}\mathfrak{S}_d$ on $V^{\otimes d}$ to an action of $\mathcal{H}_\nu(d)$ in such a way that this action commutes with the action of $\mathcal{U}_\nu(\mathfrak{gl}_n)$. The quantum analogue of Schur-Weyl duality, see [20], says

Proposition 3.5. *We have surjective homomorphisms*

$$\phi_d : \mathcal{U}_\nu(\mathfrak{gl}_n) \rightarrow \text{End}_{\mathcal{H}_\nu(d)}(V^{\otimes d}) \quad \text{and} \quad \psi_d : \mathcal{H}_\nu(d) \rightarrow \text{End}_{\mathcal{U}_\nu(\mathfrak{gl}_n)}(V^{\otimes d})^{op}$$

for all $d \geq 1$.

The image of ϕ_d in $\text{End}_{\mathcal{H}_\nu(d)}(V^{\otimes d})$ is the ν -Schur algebra, denoted $S_\nu(n, d)$. The category $\mathcal{C}_{n, \nu}$ of finite-dimensional representations of $\mathcal{U}_\nu(\mathfrak{gl}_n)$ is no longer semi-simple in general. However, the simple modules in this category are still labeled L_λ for $\lambda \in \mathbb{Z}^n$ such that $\lambda_i - \lambda_{i+1} \geq 0$ for all $1 \leq i \leq n-1$. Moreover, if we let $\mathcal{C}_{n, \nu}(d)$ denote the full subcategory of $\mathcal{C}_{n, \nu}$ consisting of all modules whose composition factors are L_λ for $\lambda \in \mathcal{P}_d(n)$ then we again have $\mathcal{C}_{n, \nu} = S_\nu(n, d)\text{-mod}$. It is known that $S_\nu(n, d)\text{-mod}$ is a highest weight category with standard modules W_λ .

3.5. Rouquier's equivalence. In order to calculate the multiplicities

$$m_{\lambda, \mu} = [\Delta(\lambda) : L(\mu)]$$

one has to make a long chain of connections and reformulations of the question and the end answer relies on several remarkable results. The first of these is Rouquier's equivalence.

Theorem 3.6. *Assume⁶ that $\mathbf{c} \in \mathbb{Q}_{\geq 0}$ does not lie in $\frac{1}{2} + \mathbb{Z}$ and set $\nu = \exp(2\pi\sqrt{-1}\mathbf{c})$. Then there is an equivalence of highest weight categories*

$$\Psi : \mathcal{O} \xrightarrow{\sim} S_\nu(n)\text{-mod},$$

such that $\Psi(\Delta(\lambda)) = W_\lambda$ and $\Psi(L(\lambda)) = L_\lambda$.

It is conjectured that the restriction $\mathbf{c} \notin \frac{1}{2} + \mathbb{Z}$ is not required. For simplicity, we will ignore this restriction and assume that Rouquier's Theorem holds for all $\mathbf{c} \in \mathbb{Q}_{\geq 0}$. Thus, to calculate $m_{\lambda, \mu}$ it suffices to try and describe the numbers $[W_\lambda : L_\mu]$.

3.6. The quantum affine enveloping algebra. Let q be an indeterminate and let I be the set $\{1, \dots, r\}$. We now turn our attention to another quantized enveloping algebra, this time of an affine Lie algebra. The *quantum affine enveloping algebra* $\mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$ is the $\mathbb{Q}(q)$ -algebra generated by $E_i, F_i, K_i^{\pm 1}$ for $i \in I$ and satisfying the relations

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, & K_i K_j &= K_j K_i, & \forall 1 \leq i, j \leq r \\ K_i E_j &= q^{a_{i,j}} E_j K_i, & K_i F_j &= q^{-a_{i,j}} F_j K_i, & \forall 1 \leq i, j \leq r \\ [E_i, F_j] &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, & \forall 1 \leq i, j \leq r \end{aligned} \tag{4}$$

and the quantum Serre relations

$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0, \quad \forall 1 \leq i \leq r \tag{5}$$

$$F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0, \quad \forall 1 \leq i \leq r \tag{6}$$

where the indices in (5) and (6) are taken modulo r so that $0 = r$ and $r+1 = 1$. In (4), $a_{i,i} = 2$, $a_{i,i\pm 1} = -1$ and 0 otherwise. In the case $r = 2$, we take $a_{i,j} = -2$ if $i \neq j$.

3.7. The q -deformed Fock space. Let \mathcal{F}_q be the *level one Fock space* for $\mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$. It is a $\mathbb{Q}(q)$ -vector space with standard basis $\{|\lambda\rangle\}$, labeled by all partitions. The action of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$ on \mathcal{F}_q is combinatorially defined. If γ is a box of the Young tableaux corresponding to the partition λ then

⁶Note that Rouquier's rational Cherednik algebra is parameterized by $h = -\mathbf{c}$.

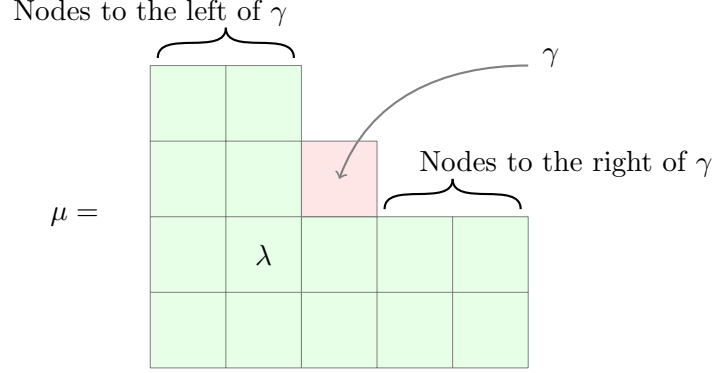


FIGURE 2. Nodes to the left and right of γ . The total partition is μ , whilst the green shaded subpartition is λ .

we say that the *residue* of γ is i , or we say that γ is an i -box of λ , if the content of γ equals i modulo r . Let λ and μ be two partitions such that μ is obtained from λ by adding a box γ with residue i ; see figure 3.7. We define

$$\begin{aligned}
 N_i(\lambda) &= |\{\text{indent } i\text{-boxes of } \lambda\}| - |\{\text{removable } i\text{-boxes of } \lambda\}|, \\
 N_i^l(\lambda, \mu) &= |\{\text{indent } i\text{-boxes of } \lambda \text{ situated to the left of } \gamma \text{ (not counting } \gamma)\}| \\
 &\quad - |\{\text{removable } i\text{-boxes of } \lambda \text{ situated to the left of } \gamma\}|, \\
 N_i^r(\lambda, \mu) &= |\{\text{indent } i\text{-boxes of } \lambda \text{ situated to the right of } \gamma \text{ (not counting } \gamma)\}| \\
 &\quad - |\{\text{removable } i\text{-boxes of } \lambda \text{ situated to the right of } \gamma\}|,
 \end{aligned}$$

Then,

$$F_i|\lambda\rangle = \sum_{\mu} q^{N_i^r(\lambda, \mu)} |\mu\rangle, \quad E_i|\mu\rangle = \sum_{\lambda} q^{N_i^l(\lambda, \mu)} |\lambda\rangle,$$

where, in each case, the sum is over all partitions such that μ/λ is a i -node, and

$$K_i|\lambda\rangle = q^{N_i(\lambda)} |\lambda\rangle.$$

See [38, Section 4.2] for further details.

Example 3.7. Let $\lambda = (5, 4, 1, 1)$ and $r = 3$ so that the Young diagram with residues of λ is

0				
1				
2	0	1	2	
0	1	2	0	1

Then

$$F_2|\lambda\rangle = |(6, 4, 1, 1)\rangle + |(5, 4, 2, 1)\rangle + q|(5, 4, 1, 1, 1)\rangle,$$

$$E_2|\lambda\rangle = q^2|(5, 3, 1, 1)\rangle,$$

and

$$K_2|\lambda\rangle = q^2|(5, 3, 1, 1)\rangle$$

Exercise 3.8. Let $\lambda = (6, 6, 3, 1, 1)$ and $r = 4$. Calculate the action of F_2 , E_4 and K_1 on $|\lambda\rangle$ (hint: draw a picture!).

Exercise 3.9. Write a program that calculates the action of the operators K_i, F_i, E_i on \mathcal{F}_q .

In fact, there is an action of a larger algebra, the quantum affine enveloping algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_r) \supset \mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$ on the space \mathcal{F}_q . The key point for us is that \mathcal{F}_q is an *irreducible* highest weight representation of $\mathcal{U}_q(\widehat{\mathfrak{gl}}_r)$, with highest weight $|\emptyset\rangle$. As a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$ -module, the Fock space is actually a direct sum of infinitely many irreducible highest weight modules, see [37].

3.8. The \mathbb{Q} -linear involution $q \mapsto \bar{q} := q^{-1}$ of $\mathbb{Q}(q)$ extends to an involution $v \mapsto \bar{v}$ of \mathcal{F}_q . In order to describe this involution, it suffices to say how to calculate $\overline{|\lambda\rangle}$. We begin by noting that a partition can also be describe as an infinite wedge as follows.

Exercise 3.10. Let \mathcal{J} be the set of strictly decreasing sequences $I = (i_1, i_2, \dots)$ such that $i_k = -k+1$ for $k \gg 0$. Show that there is a natural bijection between \mathcal{J} and \mathcal{P} , the set of all partitions. This bijection sends $\mathcal{J}_n = \{I \in \mathcal{J} \mid \sum_k (i_k + k - 1) = n\}$ to \mathcal{P}_n the set of all partitions of n .

Thus, to a partition $I \in \mathcal{J}$ we associate the infinite wedge

$$u_I = u_{i_1} \wedge u_{i_2} \wedge u_{i_3} \wedge \dots,$$

for instance

$$u_{(3^2, 2)} = u_3 \wedge u_2 \wedge u_0 \wedge u_{-3} \wedge u_{-4} \wedge \dots$$

An infinite wedge is normally ordered if it equals u_I for some $I \in \mathcal{J}$. Just as for usual wedge products, there is a “normal ordering rule” for the q -deformed wedge product. However, this normal ordering rule depends in a very nontrivial way on the integer r (and more generally, on the level l of the Fock space). In order to describe the normal ordering rule, it suffices to say how to swap two adjacent u_i ’s. Let $i < j$ be integers with $j - i \equiv m \pmod{r}$ for some $0 \leq m < r$. If $m = 0$ then

$$u_i \wedge u_j = -u_j \wedge u_i,$$

and otherwise

$$\begin{aligned} u_i \wedge u_j = & -q^{-1}u_j \wedge u_i + (q^{-2} - 1)[u_{j-m} \wedge u_{i+m} - q^{-1}u_{j-r} \wedge u_{i+r} \\ & + q^{-2}u_{j-m-r} \wedge u_{i+m+r} - q^{-3}u_{j-2r} \wedge u_{i+2r} + \dots] \end{aligned}$$

where the sum continues only as long as the terms are normally ordered. Let $I \in \mathcal{J}$ and write $\alpha_{r,k}(I)$ for the number of pairs (a, b) with $1 \leq a < b \leq k$ and $i_a - i_b \not\equiv 0 \pmod{r}$.

Proposition 3.11. *For $k \geq n$, the q -wedge*

$$\overline{u_I} = (-1)^{\binom{k}{2}} q^{\alpha_{r,k}(I)} u_{i_k} \wedge u_{i_{k-1}} \wedge \dots \wedge u_{i_1} \wedge u_{i_{k+1}} \wedge u_{i_{k+2}} \wedge \dots$$

⁷There is a typo in definition of $\alpha_{r,k}(I)$ in [39]

is independent of k .

Therefore we can define a semi-linear map, $v \mapsto \bar{v}$ on \mathcal{F}_q by

$$\overline{f(q)u_I} = f(q^{-1})\bar{u_I}.$$

This is actually an involution on \mathcal{F}_q . For $\mu \vdash n$ define

$$|\bar{\mu}\rangle = \sum_{\lambda \vdash n} a_{\lambda, \mu}(q) |\lambda\rangle.$$

Then it is known, [39, Theorem 3.3], that the polynomials $a_{\lambda, \mu}(q)$ have the following properties.

Theorem 3.12. *Let $\lambda, \mu \vdash n$.*

- (1) $a_{\lambda, \mu}(q) \in \mathbb{Z}[q, q^{-1}]$.
- (2) $a_{\lambda, \mu}(q) = 0$ unless $\lambda \triangleleft \mu$ and λ, μ have the same r -core.
- (3) $a_{\lambda, \lambda}(q) = 1$.
- (4) $a_{\lambda, \mu}(q) = a_{\mu', \lambda'}(q)$.

Example 3.13. If $r = 3$ then

$$\overline{|(4, 3, 1)\rangle} = |(4, 3, 1)\rangle + (q - q^{-1})|(3, 3, 1, 1)\rangle + (-1 + q^{-2})|(2, 2, 2, 2)\rangle + (q^2 - 1)|(2, 1, 1, 1, 1, 1)\rangle.$$

Leclerc and Thibon showed:

Theorem 3.14 ([39], Theorem 4.1). *There exist canonical basis $\{\mathcal{G}^+(\lambda)\}$ and $\{\mathcal{G}^-(\lambda)\}$ characterized by*

- (1) $\overline{\mathcal{G}^+(\lambda)} = \mathcal{G}^+(\lambda)$, $\overline{\mathcal{G}^-(\lambda)} = \mathcal{G}^-(\lambda)$.
- (2) $\mathcal{G}^+(\lambda) \equiv |\lambda\rangle \pmod{q\mathbb{Z}[q]}$ and $\mathcal{G}^-(\lambda) \equiv |\lambda\rangle \pmod{q^{-1}\mathbb{Z}[q^{-1}]}$.

It seems that the above result has nothing to do with the fact that \mathcal{F}_q is a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_r)$ -module. All that is required is that there is some involution defined on the space. However, for an arbitrary involution there is no reason to expect a canonical basis to exist (and, indeed, one can check with small examples that it does not). Of course, the involution used by Leclerc and Thibon is not arbitrary. There is a natural involution on the algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}_r)$. Since \mathcal{F}_q is irreducible as a $\mathcal{U}_q(\widehat{\mathfrak{gl}}_r)$ -module, there is a unique involution on \mathcal{F}_q such that $\overline{F|\emptyset\rangle} = \overline{F}|\emptyset\rangle$ for all $F \in \mathcal{U}_q(\widehat{\mathfrak{gl}}_r)$. It is this involution that Leclerc and Thibon use, though as we have seen they are able to give an explicit definition of this involution. Then, it follows from a general result by Lusztig, [40, §7.10], that an irreducible, highest weight module equipped with this involution admits a canonical basis. Set

$$\mathcal{G}^+(\mu) = \sum_{\lambda} d_{\lambda, \mu}(q) |\lambda\rangle, \quad \mathcal{G}^-(\lambda) = \sum_{\mu} e_{\lambda, \mu}(q) |\mu\rangle.$$

The polynomials $d_{\lambda, \mu}$ and $e_{\lambda, \mu}$ have the following properties:

- They are non-zero only if λ and μ have the same r -core, $d_{\lambda, \lambda}(q) = e_{\lambda, \lambda}(q) = 1$;
- $d_{\lambda, \mu}(q) = 0$ unless $\lambda \leq \mu$, and $e_{\lambda, \mu}(q) = 0$ unless $\mu \leq \lambda$.

3.9. GAP. In order to calculate the polynomial $\mathcal{G}^+(\mu)$ we use the computer package GAP. The file⁸ `Canonical.gap` contains the functions

⁸Available from <http://www.maths.gla.ac.uk/~gbellamy/MSRI.html>.

```
APolynomial(lambda,mu,r)
```

and

```
DPolynomial(lambda,mu,r)
```

which, when given a pair of partitions and an integer r , returns the polynomials $a_{\lambda,\mu}(q)$ and $d_{\lambda,\mu}(q)$ respectively. For example:

```
gap>Read("Canonical.gap");
gap>APolynomial([2,1,1],[3,1],2);
q^2-1
gap>DPolynomial([1,1,1,1,1],[5],2);
q^2
gap>
```

Exercise 3.15. For $r = 2$, describe $\mathcal{G}(\lambda)$ for all $\lambda \vdash 4$ and $\lambda \vdash 5$.

Exercise 3.16. (Harder) Write a program that calculates the polynomials $e_{\lambda,\mu}(q)$.

3.10. Then, assuming that $r > 1$, [51, Theorem 11] says that

$$[W_\lambda : L_\mu] = d_{\lambda',\mu'}(1), \quad [L_\lambda : W_\mu] = e_{\lambda,\mu}(1).$$

Combining the results of [39], [51] and [45]:

Theorem 3.17 (Leclerc-Thibon, Vasserot-Varagnolo, Rouquier). *We have*

$$[\Delta(\lambda) : L(\mu)] = d_{\lambda',\mu'}(1), \quad \text{and} \quad [L(\lambda) : \Delta(\mu)] = e_{\lambda,\mu}(1). \quad (7)$$

Theorem 3.17 gives us a way to express the graded character of the simple modules $L(\lambda)$ in terms of the numbers $e_{\lambda,\mu}(1)$.

Corollary 3.18. *We have*

$$\text{ch}(L(\lambda)) = \frac{1}{(1-t)^n} \cdot \left(\sum_{\mu \leq \lambda} e_{\lambda,\mu}(1) \dim(\mu) t^{c_\mu} \right). \quad (8)$$

Proof. The corollary depends on two key facts about standard modules. Firstly, they form a \mathbb{Z} -basis of the Grothendieck ring $K_0(\mathcal{O})$ (exercise 2.20) and, secondly, it is easy to calculate the character of $\Delta(\lambda)$. Theorem 3.17 implies that we have

$$[L(\lambda)] = \sum_{\mu \leq \lambda} e_{\lambda,\mu}(1) [\Delta(\mu)]$$

in $K_0(\mathcal{O})$. Now the corollary follows from the fact that $\text{ch}(\Delta(\mu)) = \frac{\dim(\mu)t^{c_\mu}}{(1-t)^n}$. □

Notice that, though equation (8) looks very simple, it is extremely difficult to extract meaningful information from it. For instance, one cannot tell whether $L(\lambda)$ is finite dimensional by looking at this character. Also, note that the coefficients $e_{\lambda,\mu}(1)$ are often negative so the numerator is a polynomial with some negative coefficients. But expanding the fraction as a power-series

around zero gives a series with *only positive* integer coefficients - all negative coefficients magically disappear!

Example 3.19. Let $n = 4$ and $\mathbf{c} = \frac{5}{3}$. In this case, the numbers $e_{\lambda,\mu}(1)$ and \mathbf{c}_λ are:

$\mu \backslash \lambda$	(4)	(3, 1)	(2, 2)	(2, 1, 1)	(1, 1, 1, 1)
(4)	1	0	0	0	0
(3, 1)	0	1	0	0	0
(2, 2)	-1	0	1	0	0
(2, 1, 1)	0	0	0	1	0
(1, 1, 1, 1)	1	0	-1	0	1
\mathbf{c}_λ	-8	$-\frac{4}{3}$	2	$\frac{16}{3}$	12

If we define $\text{ch}_\lambda(t) := (1 - t)^4 \cdot \text{ch}(L(\lambda))$, then

$$\begin{aligned} \text{ch}_{(4)}(t) &= t^{-8} - 2t^2 + t^{12}, & \text{ch}_{(3,1)}(t) &= 3t^{-\frac{4}{3}}, \\ \text{ch}_{(2,2)}(t) &= 2t^2 - t^{12}, & \text{ch}_{(2,1,1)}(t) &= 3t^{\frac{16}{3}}, & \text{ch}_{(1,1,1,1)}(t) &= t^{12}. \end{aligned}$$

When $\mathbf{c} = \frac{9}{4}$ we get:

$\mu \backslash \lambda$	(4)	(3, 1)	(2, 2)	(2, 1, 1)	(1, 1, 1, 1)
(4)	1	0	0	0	0
(3, 1)	-1	1	0	0	0
(2, 2)	0	0	1	0	0
(2, 1, 1)	1	-1	0	1	0
(1, 1, 1, 1)	-1	1	0	-1	1
\mathbf{c}_λ	$-\frac{23}{2}$	$-\frac{5}{2}$	2	$\frac{13}{2}$	$\frac{31}{2}$

and

$$\begin{aligned} \text{ch}_{(4)}(t) &= t^{-\frac{23}{2}} - 3t^{-\frac{5}{2}} + 3t^{\frac{13}{2}} - t^{\frac{31}{2}}, & \text{ch}_{(3,1)}(t) &= 3t^{-\frac{5}{2}} - 3t^{\frac{13}{2}} + t^{\frac{31}{2}}, \\ \text{ch}_{(2,2)}(t) &= 2t^2, & \text{ch}_{(2,1,1)}(t) &= 3t^{\frac{13}{2}} - t^{\frac{31}{2}}, & \text{ch}_{(1,1,1,1)}(t) &= t^{\frac{31}{2}}. \end{aligned}$$

Let $\mathfrak{h} \subset \mathbb{C}^4$ be the three dimensional reflection representation. One could also consider representations of $\mathbf{H}_{\mathbf{c}}(\mathfrak{h}, \mathfrak{S}_4)$ instead of $\mathbf{H}_{\mathbf{c}}(\mathbb{C}^4, \mathfrak{S}_4)$. If $\mathbb{C}^4 = \mathfrak{h} \oplus \mathbb{C}$ as a \mathfrak{S}_4 -module, then $\mathbf{H}_{\mathbf{c}}(\mathbb{C}^4, \mathfrak{S}_4) = \mathbf{H}_{\mathbf{c}}(\mathfrak{h}, \mathfrak{S}_4) \otimes \mathcal{D}(\mathbb{C})$, which implies that $\Delta_{\mathbb{C}^4}(\lambda) = \Delta_{\mathfrak{h}}(\lambda) \otimes \mathbb{C}[x]$ for each $\lambda \vdash 4$. Then,

$$\text{ch}(L_{\mathbb{C}^4}(\lambda)) = \frac{1}{1-t} \cdot \text{ch}(L_{\mathfrak{h}}(\lambda)),$$

and hence $\text{ch}_\lambda(t) = (1-t)^3 \cdot \text{ch}(L_{\mathfrak{h}}(\lambda))$. If we consider $\lambda = (4)$ with $\mathbf{c} = \frac{9}{4}$ then notice that

$$\frac{t^{-\frac{23}{2}} - 3t^{-\frac{5}{2}} + 3t^{\frac{13}{2}} - t^{\frac{31}{2}}}{(1-t)^3} = t^{-\frac{23}{2}} \left[\frac{1 - 3t^9 + 3t^{18} - t^{27}}{(1-t)^3} \right] = t^{-\frac{23}{2}} (t^{24} + 3t^{23} + 6t^{22} + \dots + 3t + 1).$$

This implies that $L((4))$ is finite dimensional. Evaluating the above polynomial shows that it actually has dimension 729.

Exercise 3.20. (1) For $c = \frac{7}{2}$, compute the character of all the simple modules of category \mathcal{O} for $\mathbf{H}_{\mathbf{c}}(\mathfrak{S}_5)$.

- (2) For $c = \frac{11}{3}$, compute the character of all the simple modules of category \mathcal{O} for $H_c(\mathfrak{S}_4)$.
- (3) What are the blocks of \mathcal{O} for \mathfrak{S}_5 at $c = \frac{7}{2}$? at $c = \frac{103}{3}$ or at $c = \frac{29}{5}$? Hint: It is known that two partitions are in the same block of the q -Schur algebra if and only if they have the same r -core.

Since the action of \mathbf{eu} on a module $M \in \mathcal{O}$ commutes with the action of W , the multiplicity space of a representation $\lambda \in \text{Irr}(W)$ is a \mathbf{eu} -module. Therefore, one can refine the character ch to

$$\text{ch}_W(M) = \sum_{\lambda} \text{ch}(M(\lambda))[\lambda],$$

where $M = \bigoplus_{\lambda \in \text{Irr}(W)} M(\lambda) \otimes \lambda$ as a W -module. Given $\lambda, \mu \in \text{Irr}(W)$, we define the *generalized fake polynomial* $f_{\lambda, \mu}(t)$ by

$$f_{\lambda, \mu}(t) = \sum_{i \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}]_i^{\text{co}W} \otimes \lambda : \mu] t^i,$$

a Laurent polynomial.

Exercise 3.21. What is the \mathfrak{S}_4 -graded character of the simple modules, expressed in terms of generalized fake polynomials, in \mathcal{O} when $c = \frac{5}{2}$?

3.11. Yvonne's conjecture. In many ways, the most interesting, and hence most intensely studied, class of rational Cherednik algebras are those associated to the complex reflection group

$$W = \mathfrak{S}_n \wr \mathbb{Z}_l = \mathfrak{S}_n \ltimes \mathbb{Z}_l^n,$$

the wreath product of the symmetric group with the cyclic group of order l . Fix ζ a primitive l -th root of unity. The reflection representation for $\mathfrak{S}_n \wr \mathbb{Z}_l$ is \mathbb{C}^n . If y_1, \dots, y_n is the standard basis of \mathbb{C}^n then

$$\mathbb{Z}_l^n = \{g_i^j \mid 1 \leq i \leq n, 0 \leq j \leq l-1\}$$

and

$$g_i^j \cdot y_k = \begin{cases} \zeta^j y_k & \text{if } k = i \\ y_k & \text{otherwise} \end{cases}$$

The symmetric group acts as $\sigma \cdot y_i = y_{\sigma(i)}$. This implies that

$$\sigma \cdot g_i = g_{\sigma(i)} \cdot \sigma.$$

The conjugacy classes of reflections in W are

$$R = \{s_{i,j} g_i g_j^{-1} \mid 1 \leq i < j \leq n\}, \quad S_i = \{g_j^i \mid 1 \leq j \leq n\}, \quad 1 \leq i \leq l-1.$$

and we define \mathbf{c} by $\mathbf{c}(s_{i,j}) = c$, $\mathbf{c}(g_j^i) = c_i$ so that the rational Cherednik algebra is parameterized by c, c_1, \dots, c_{l-1} . To relate category \mathcal{O} to a certain Fock space, we introduce new parameters $\mathbf{h} = (h, h_0, \dots, h_{l-1})$, where

$$c = 2h, \quad c_i = \sum_{j=0}^{l-1} \zeta^{-ij} (h_j - h_{j+1}), \quad \forall 1 \leq i \leq l-1.$$

Note that the above equations do not uniquely specify h_0, \dots, h_{l-1} ; one can choose $h_0 + \dots + h_{l-1}$ freely. Finally, we define $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$ and $e \in \mathbb{N}$ by $h = \frac{1}{e}$ and

$$h_j = \frac{s_j}{e} - \frac{j}{d}.$$

The irreducible representations of $\mathfrak{S}_n \wr \mathbb{Z}_l$ are labeled by l -multipartitions of n , where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ is an l -tuple of partitions $\lambda^{(i)}$ such that

$$\sum_{i=1}^l |\lambda^{(i)}| = n.$$

The set of all l -multipartitions is denoted \mathcal{P}^l and the subset of all l -multipartitions of n is denoted \mathcal{P}_n^l . Thus, the simple modules in category \mathcal{O} are $L(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^l$. It is actually quite easy to construct the simple W -modules $\boldsymbol{\lambda}$ once one has constructed all representations of the symmetric group.

In [49], Uglov define for each $l \geq 1$ and $\mathbf{s} \in \mathbb{Z}^l$ a level l Fock space $\mathcal{F}_q^l[\mathbf{s}]$ with multi-charge \mathbf{s} . This is again a representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_r)$, this time with $\mathbb{Q}(q)$ -basis $|\boldsymbol{\lambda}\rangle$ given by l -multi-partitions. The action of the operators F_i, E_i and K_i have a similar combinatorial flavour as for $l = 1$. The space $\mathcal{F}_q^l[\mathbf{s}]$ is also equipped with a \mathbb{Q} -linear involution.

Theorem 3.22 ([49], Theorem 2.5). *There exists a unique $\mathbb{Q}(q)$ -basis $\mathcal{G}(\boldsymbol{\lambda})$ of $\mathcal{F}_q^l[\mathbf{s}]$ such that*

- (1) $\overline{\mathcal{G}(\boldsymbol{\lambda})} = \mathcal{G}(\boldsymbol{\lambda})$.
- (2) $\mathcal{G}(\boldsymbol{\lambda}) - \boldsymbol{\lambda} \in \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_n^l} q\mathbb{Q}[q]|\boldsymbol{\mu}\rangle$ if $\boldsymbol{\lambda} \vdash n$.

Therefore, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^l$, we define $d_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(q)$ by

$$\mathcal{G}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n^l} d_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(q) |\boldsymbol{\lambda}\rangle.$$

The following conjecture, originally due to Yvonne, [56, Conjecture 2.13], but in the generality stated here due to Rouquier, [45, Section 6.5], relates the multiplicities of simple modules inside standard modules for $H_c(\mathfrak{S}_n \wr \mathbb{Z}_l)$ to the polynomials $d_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(q)$.

Conjecture 3.23. For all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^l$ we have

$$[\Delta(\boldsymbol{\lambda}) : L(\boldsymbol{\mu})] = d_{\boldsymbol{\lambda}^\dagger, \boldsymbol{\mu}^\dagger}(1).$$

If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is an m -multi-partition then

$$\boldsymbol{\lambda}^\dagger = ((\lambda^{(m)})', \dots, (\lambda^{(1)})'),$$

where $(\lambda^{(i)})'$ is the usual transpose of the partition $\lambda^{(i)}$.

3.12. Additional remark.

- Theorem 3.6 is given in [45, Theorem 6.11]. Its proof is based on the uniqueness of quasi-hereditary covers of the Hecke algebra, as shown in [45].

4. THE KZ FUNCTOR

Recall from the first lecture that we have the Dunkl embedding $H_c(W) \hookrightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$, which becomes an isomorphism

$$H_c(W)[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$$

after localization. If M is a module in category \mathcal{O} then we can also localize M to get $M[\delta^{-1}]$, a $\mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W$ -module. In particular, $M[\delta^{-1}]$ is a $\mathcal{D}(\mathfrak{h}_{\text{reg}})$ -module. Therefore, we can use some of the very powerful machinery in the theory of \mathcal{D} -modules, such as the Riemann-Hilbert correspondence, to study $M[\delta^{-1}]$. This is the basic idea behind the KZ-functor. The goal of this lecture is to try and describe in some geometric way the image of category \mathcal{O} under this localization procedure. Remarkably, what one gets in the end is a functor

$$\text{KZ} : \mathcal{O} \longrightarrow \mathcal{H}_{\mathbf{q}}(W)\text{-mod},$$

from category \mathcal{O} to the category of finitely generated modules over the *Hecke algebra* $\mathcal{H}_{\mathbf{q}}(W)$ associated to W .

4.1. Integrable connections. A $\mathcal{D}(\mathfrak{h}_{\text{reg}})$ -module which is finitely generated as a $\mathbb{C}[\mathfrak{h}_{\text{reg}}]$ -module is called an *integrable connection*. It is a classical result that every integrable connection is actually free as a $\mathbb{C}[\mathfrak{h}_{\text{reg}}]$ -module. Therefore, if N is an integrable connection,

$$N \simeq \bigoplus_{i=1}^k \mathbb{C}[\mathfrak{h}_{\text{reg}}] u_i$$

for some $u_i \in N$. If we fix coordinates x_1, \dots, x_n such that $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \dots, x_n]$ and let $\partial_i = \frac{\partial}{\partial x_i}$, then the action of $\mathcal{D}(\mathfrak{h}_{\text{reg}})$ on N is then completely encoded in the equations

$$\partial_l \cdot u_i = \sum_{j=1}^k f_{l,i}^j u_j, \tag{9}$$

for some polynomials⁹ $f_{l,i}^k \in \mathbb{C}[\mathfrak{h}_{\text{reg}}]$. The integer k is called the *rank* of N .

A natural approach to studying integrable connections is to look at their space of solutions. Since very few differential equations have polynomial solutions, this approach only makes sense in the analytic topology. So we'll write $\mathfrak{h}_{\text{reg}}^{\text{an}}$ for the same space, but now equipped with the analytic topology and $\mathbb{C}[\mathfrak{h}_{\text{reg}}^{\text{an}}]$ denotes the ring of *holomorphic* functions on $\mathfrak{h}_{\text{reg}}^{\text{an}}$. Since $\mathcal{D}(\mathfrak{h}_{\text{reg}}^{\text{an}}) = \mathbb{C}[\mathfrak{h}_{\text{reg}}^{\text{an}}] \otimes_{\mathbb{C}[\mathfrak{h}_{\text{reg}}]} \mathcal{D}(\mathfrak{h}_{\text{reg}})$, we have a natural functor

$$\mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W\text{-mod} \rightarrow \mathcal{D}(\mathfrak{h}_{\text{reg}}^{\text{an}}) \rtimes W\text{-mod}, \quad M \mapsto M^{\text{an}} := \mathbb{C}[\mathfrak{h}_{\text{reg}}^{\text{an}}] \otimes_{\mathbb{C}[\mathfrak{h}_{\text{reg}}]} M.$$

Since $\mathbb{C}[\mathfrak{h}_{\text{reg}}^{\text{an}}]$ is faithfully flat, this is exact and conservative i.e. $M^{\text{an}} = 0$ implies that $M = 0$. On any simply connected open subset U of $\mathfrak{h}_{\text{reg}}^{\text{an}}$, the vector space

$$\text{Hom}_{\mathcal{D}(\mathfrak{h}_{\text{reg}}^{\text{an}})}(N^{\text{an}}, \mathbb{C}[U])$$

⁹The condition $[\partial_l, \partial_m] = 0$ implies that one cannot choose arbitrary polynomials $f_{i,i}^j$.

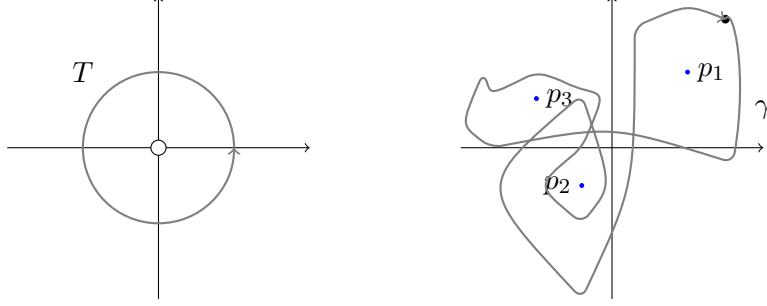


FIGURE 3. The generator T of the braid group $\pi_1(\mathbb{C}^\times)$, and non-trivial element γ in $\pi_1(\mathbb{C} \setminus \{p_1, p_2, p_3\}, \bullet)$.

is k -dimensional because it is the space of solutions of a $k \times k$ matrix of first order linear differential equations. These spaces of local solutions glue together to give a rank k *local system* $\text{Sol}(N)$ on $\mathfrak{h}_{\text{reg}}$ i.e. a locally constant sheaf of \mathbb{C} -vector spaces such that each fiber has dimension k , see figure 4.2 for an illustration. The reader should have in mind the idea of analytic continuation from complex analysis, of which the notion of local system is a generalization.

4.2. Regular singularities. Assume now that $\dim \mathfrak{h} = 1$, so that $\mathfrak{h} = \mathbb{C}$ and $\mathfrak{h}_{\text{reg}} = \mathbb{C}^\times$. Then, we say that N is a *regular connection* (or has *regular singularities*) if, with respect to some $\mathbb{C}[\mathfrak{h}_{\text{reg}}^{\text{an}}]$ -basis of N^{an} , equation (9) becomes

$$\partial \cdot u_i = \sum_{j=1}^k \frac{a_{i,j}}{x} u_j, \quad a_{i,j} \in \mathbb{C}. \quad (10)$$

When $\dim \mathfrak{h} > 1$, we say that N has regular singularities if the restriction $N|_C$ of N to any smooth curve $C \subset \mathfrak{h}_{\text{reg}}$ has, after a suitable change of basis, the form (10). There are two natural functors from the category $\text{Conn}^{\text{reg}}(\mathfrak{h}_{\text{reg}})$ of integrable connections with regular singularities on $\mathfrak{h}_{\text{reg}}$ to local systems on $\mathfrak{h}_{\text{reg}}$. Firstly, there is a *solutions functor*

$$N \mapsto \text{Sol}(N) := \mathcal{H}om_{\mathcal{D}_{\mathfrak{h}_{\text{reg}}^{\text{an}}}}(N^{\text{an}}, \mathcal{O}_{\mathfrak{h}_{\text{reg}}^{\text{an}}}^{\text{an}}),$$

and secondly the *deRham functor*

$$N \mapsto \text{DR}(N) := \mathcal{H}om_{\mathcal{D}_{\mathfrak{h}_{\text{reg}}^{\text{an}}}}(\mathcal{O}_{\mathfrak{h}_{\text{reg}}^{\text{an}}}^{\text{an}}, N^{\text{an}}).$$

There is a natural duality on integrable connection which intertwines the solutions and deRham functors, see [36, Chapter 7] for details. Sticking with conventions, we'll work with the deRham functor. Note that there is a natural identification

$$\text{DR}(N) = \{n \in N^{\text{an}} \mid \partial_i \cdot n = 0 \ \forall i\} =: N^\nabla,$$

where N^∇ is usually referred to as the subsheaf of *horizontal sections* of N^{an} . Recall that there is a natural (up to a choice of base point) equivalence between the category of local systems on $\mathfrak{h}_{\text{reg}}^{\text{an}}$ and the category $\pi_1(\mathfrak{h}_{\text{reg}})$ -mod of finite-dimensional representations of the fundamental group $\pi_1(\mathfrak{h}_{\text{reg}}^{\text{an}})$ of $\mathfrak{h}_{\text{reg}}^{\text{an}}$.

4.3. Deligne's version of the Riemann-Hilbert correspondence, [18], says that:

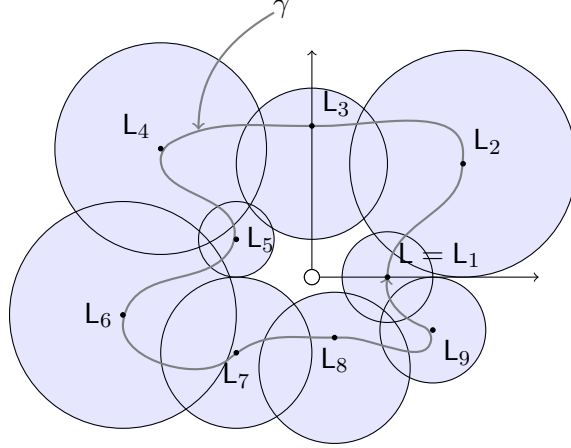


FIGURE 4. The locally constant sheaf (local system) L on \mathbb{C}^\times . On each open set, the local system L_i is constant, and on overlaps we have $L_i \simeq L_{i+1}$. Composing these isomorphisms defines an automorphism $L \xrightarrow{\sim} L$, which is the action of the path γ on the stalk of L at 1.

Theorem 4.1. *The deRham and solutions functors*

$$\mathrm{DR}, \mathrm{Sol} : \mathrm{Conn}^{\mathrm{reg}}(\mathfrak{h}_{\mathrm{reg}}) \rightarrow \pi_1(\mathfrak{h}_{\mathrm{reg}})\text{-mod}$$

are equivalences.

Why is the notion of regular connection crucial in Deligne’s version of the Riemann-Hilbert correspondence? A complete answer to this question is beyond the authors understanding. But one important point is that there are, in general, many non-isomorphic integrable connection that will give rise to the same local system. So what Deligne showed is that, for a given local system L , the notion of regular connection gives one a canonical representative in the set of all connections whose solutions are L .

4.4. Equivariance. The group W acts freely on $\mathfrak{h}_{\mathrm{reg}}$. Therefore, there is a natural isomorphism $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W \simeq \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$. If the $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ -module N is actually a $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$ -module, or a “ W -equivariant \mathcal{D} -module”, then N descends to a \mathcal{D} -module on the quotient space $\mathfrak{h}_{\mathrm{reg}}/W$. Concretely, this just means that the identification $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W \simeq \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$ allows us to think of N^W as a $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$ -module¹⁰. Therefore, we may define the KZ-functor $\mathrm{KZ} : \mathcal{O} \rightarrow \mathbb{C}\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)\text{-mod}$ by

$$\mathrm{KZ}(M) := \mathrm{DR}(M[\delta^{-1}]^W) = (((M[\delta^{-1}])^W)^{an})^\nabla.$$

¹⁰Equivalently, there is a (unique up to isomorphism) $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}}/W)$ -module N' such that $N \simeq \mathbb{C}[\mathfrak{h}_{\mathrm{reg}}] \otimes_{\mathbb{C}[\mathfrak{h}_{\mathrm{reg}}]^W} N'$ as $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ -modules.

The following diagram should help the reader unpack the definition of the KZ-functor.

$$\begin{array}{ccc}
\mathcal{O} & \xrightarrow{(-)[\delta^{-1}]} & \mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes W\text{-mod} \\
& \searrow & \downarrow (-)^W \wr \\
& & \mathcal{D}(\mathfrak{h}_{\text{reg}}/W)\text{-mod} \xrightarrow{(-)^{an}} \mathcal{D}(\mathfrak{h}_{\text{reg}}^{an}/W)\text{-mod} \\
& \searrow \text{KZ} & \downarrow (-)^\nabla \\
& & \mathbb{C}\pi_1(\mathfrak{h}_{\text{reg}}/W)\text{-mod.}
\end{array} \tag{11}$$

Exercise 4.2. By considering the case of \mathbb{Z}_2 , show that the natural map $\mathcal{D}(\mathfrak{h})^W \rightarrow \mathcal{D}(\mathfrak{h}/W)$ is not an isomorphism. Which of injectivity or surjectivity fails? Hint: for complete rigor, consider the associated graded map $\text{gr } \mathcal{D}(\mathfrak{h})^W \rightarrow \text{gr } \mathcal{D}(\mathfrak{h}/W)$.

4.5. A change of parameters. In order to relate, in the next section, the rational Cherednik algebra $\mathbf{H}_{\mathbf{c}}(W)$, via the KZ-functor, with the cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$, we need to change the way we parameterize $\mathbf{H}_{\mathbf{c}}(W)$. Each complex reflection $s \in \mathcal{S}$ defines a reflecting hyperplane $H = \ker \alpha_s \subset \mathfrak{h}$. Let \mathcal{A} denote the set of all hyperplanes arising this way. For a given $H \in \mathcal{A}$, the subgroup $W_H = \{w \in W \mid w(H) \subset H\}$ of W is cyclic. Let $W_H^* = W_H \setminus \{1\}$. Then

$$\mathcal{S} = \bigcup_{H \in \mathcal{A}} W_H^*.$$

We may, without loss of generality, assume that

$$\alpha_H := \alpha_s = \alpha_{s'}, \quad \alpha_H^\vee := \alpha_s^\vee = \alpha_{s'}^\vee, \quad \forall s, s' \in W_H^*.$$

Then the relation (2) becomes

$$[y, x] = (y, x) - \sum_{H \in \mathcal{A}} (y, \alpha_H)(\alpha_H^\vee, x) \left(\sum_{s \in W_H^*} \mathbf{c}(s)s \right), \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}. \tag{12}$$

Let $n_H = |W_H|$ and $e_{H,i} = \frac{1}{n_H} \sum_{w \in W_H} (\det w)^i w$ for $0 \leq i \leq n_H - 1$ be the primitive idempotents in $\mathbb{C}W_H$. Define $k_{H,i} \in \mathbb{C}$ for $H \in \mathcal{A}$ and $0 \leq i \leq n_H - 1$ by

$$\sum_{s \in W_H^*} \mathbf{c}(s)s = n_H \sum_{i=0}^{n_H-1} (k_{H,i+1} - k_{H,i})e_{H,i},$$

and $k_{H,0} = k_{H,n_H} = 0$. Note that this forces $k_{H,i} = k_{w(H),i}$ for all $w \in W$ and $H \in \mathcal{A}$. Thus, the parameters $k_{H,i}$ are W -invariant. This implies that

$$\mathbf{c}(s) = \sum_{i=0}^{n_H-1} (\det s)^i (k_{H,i+1} - k_{H,i}),$$

and the relation (12) becomes

$$[y, x] = (y, x) - \sum_{H \in \mathcal{A}} (y, \alpha_H)(\alpha_H^\vee, x) n_H \sum_{i=0}^{n_H-1} (k_{H,i+1} - k_{H,i})e_{H,i}, \quad \forall x \in \mathfrak{h}^*, y \in \mathfrak{h}. \tag{13}$$

Therefore $H_c(W) = H_k(W)$, where

$$\mathbf{k} = \{k_{H,i} \mid H \in \mathcal{A}, 0 \leq i \leq n_H - 1, k_{H,0} = k_{H,n_H} = 0 \text{ and } k_{H,i} = k_{w(H),i} \ \forall w, H, i\}.$$

4.6. The cyclotomic Hecke algebra. The braid group $\pi_1(\mathfrak{h}_{\text{reg}}/W)$ has generators $\{T_s \mid s \in \mathcal{S}\}$, where T_s is an s -generator of the monodromy around H ; see [8, Section 4.C] for the precise definition. The T_s satisfy certain “braid relations”. Fix $\mathbf{q} = \{q_{H,i} \in \mathbb{C}^\times \mid H \in \mathcal{A}, 0 \leq i \leq n_H - 1\}$. The *cyclotomic Hecke algebra* $\mathcal{H}_{\mathbf{q}}(W)$ is the quotient of the group algebra $\mathbb{C}\pi_1(\mathfrak{h}_{\text{reg}}/W)$ by the two-sided ideal generated by

$$\prod_{i=0}^{n_H-1} (T_s - q_{H,i}), \quad \forall s \in \mathcal{S},$$

where H is the hyperplane defined by s .

Example 4.3. In type A the Hecke algebra is the algebra generated by T_1, \dots, T_{n-1} , satisfying the braid relations

$$\begin{aligned} T_i T_j &= T_j T_i, \quad \forall |i - j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \forall 1 \leq i \leq n - 2 \end{aligned}$$

and the additional relation

$$(T_i - \mathbf{q})(T_i + 1) = 0 \quad \forall 1 \leq i \leq n - 1.$$

For each $H \in \mathcal{A}$, fix a generator s_H of W_H . Given a parameter \mathbf{k} for the rational Cherednik algebra, define \mathbf{q} by

$$q_{H,i} = (\det s_H)^{-i} \exp(2\pi\sqrt{-1}k_{H,i}).$$

Based on [8, Theorem 4.12], the following key result was proved in [27, Theorem 5.13].

Theorem 4.4. *The KZ-functor factors through $\mathcal{H}_{\mathbf{q}}(W)$ -mod.*

Since each of the functors appearing in diagram 11 is exact, the KZ-functor is an exact functor. Therefore, by [43, Theorem ??], there exists a projective module $P_{KZ} \in \mathcal{O}$ such that

$$\text{KZ}(-) \simeq \text{Hom}_{\mathcal{O}}(P_{KZ}, -). \tag{14}$$

Then, Theorem 4.4 implies that P_{KZ} is a $(H_c(W), \mathcal{H}_{\mathbf{q}}(W))$ -bimodule and the action of $\mathcal{H}_{\mathbf{q}}(W)$ on the right of P_{KZ} defines an algebra morphism

$$\phi : \mathcal{H}_{\mathbf{q}}(W) \longrightarrow \text{End}_{H_c(W)}(P_{KZ})^{op}.$$

Lemma 4.5. *Let \mathcal{A} be an abelian, Artinian category and \mathcal{A}' a full subcategory, closed under quotients. Let $F : \mathcal{A}' \rightarrow \mathcal{A}$ be the inclusion functor. Define ${}^\perp F : \mathcal{A} \rightarrow \mathcal{A}'$ by setting ${}^\perp F(M)$ to be the largest quotient of M contained in \mathcal{A}' . Then ${}^\perp F$ is left adjoint to F and the adjunction $\eta : \text{id}_{\mathcal{A}} \rightarrow F \circ ({}^\perp F)$ is surjective.*

Proof. We begin by showing that ${}^\perp F$ is well-defined. We need to show that, for each $M \in \mathcal{A}$, there is a unique maximal quotient N of M contained in \mathcal{A}' . Let

$$K = \{N' \subseteq M \mid M/N' \in \mathcal{A}'\}.$$

Note that if N'_1 and N'_2 belong to K then $N'_1 \cap N'_2$ belongs to K . Therefore, if $N'_1 \in K$ is not contained in all other $N' \in K$, we choose $N' \in K$ such that $N'_1 \not\subseteq N'$ and set $N'_2 = N' \cap N'_1 \subsetneq N'_1$. Continuing this way we construct a descending chain of submodules $N'_1 \supsetneq N'_2 \supsetneq \cdots$ of M . Since \mathcal{A} is assumed to be Artinian, this chain must eventually stop. Hence, there is a unique minimal element under inclusion in K . It is clear that ${}^\perp F$ is left adjoint to F and the adjunction η just sends M to the maximal quotient of M in \mathcal{A}' , hence is surjective. \square

Theorem 4.6 (Double centralizer theorem). *We have an isomorphism*

$$\phi : \mathcal{H}_{\mathbf{q}}(W) \xrightarrow{\sim} \text{End}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{\text{KZ}})^{op}.$$

Proof. Let \mathcal{A} denote the image of the KZ functor in $\mathcal{H}_{\mathbf{q}}(W)\text{-mod}$. It is a full subcategory of $\mathcal{H}_{\mathbf{q}}(W)\text{-mod}$. It is also closed under quotients. To see this, notice that it suffices to show that the image of \mathcal{O} under the localization functor $(-)[\delta^{-1}]$ is closed under quotients. If N is a non-zero quotient of $M[\delta^{-1}]$ for some $M \in \mathcal{O}$, then it is easy to check that the image N' of M under $M \rightarrow M[\delta^{-1}] \twoheadrightarrow N$ is non-zero and generates N . The claim follows. Then, Lemma 4.5 implies that ϕ is surjective. Hence to show that it is an isomorphism, it suffices to calculate the dimension of $\text{End}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{\text{KZ}})$. Note that

$$P_{\text{KZ}} = \bigoplus_{\lambda \in \text{Irr}(W)} \dim \text{KZ}(L(\lambda)) P(\lambda)$$

(prove this!). Hence, using BGG reciprocity, we have

$$\begin{aligned} \dim \text{End}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{\text{KZ}}) &= \bigoplus_{\lambda, \mu} \dim \text{KZ}(L(\lambda)) \dim \text{KZ}(L(\mu)) \text{Hom}_{\mathcal{O}}(P(\lambda), P(\mu)) \\ &= \bigoplus_{\lambda, \mu} \dim \text{KZ}(L(\lambda)) \dim \text{KZ}(L(\mu)) [P(\mu) : L(\lambda)] \\ &= \bigoplus_{\lambda, \mu, \nu} \dim \text{KZ}(L(\lambda)) \dim \text{KZ}(L(\mu)) [P(\mu) : \Delta(\nu)] [\Delta(\nu) : L(\lambda)] \\ &= \bigoplus_{\lambda, \mu, \nu} \dim \text{KZ}(L(\lambda)) \dim \text{KZ}(L(\mu)) [\Delta(\nu) : L(\mu)] [\Delta(\nu) : L(\lambda)] \\ &= \bigoplus_{\nu} (\dim \text{KZ}(\Delta(\nu)))^2 \end{aligned}$$

Since $\Delta(\nu)$ is a free $\mathbb{C}[\hbar]$ -module of rank $\dim(\nu)$, its localization $\Delta(\nu)[\delta^{-1}]$ is an integrable connection of rank $\dim(\nu)$. Hence, $\dim \text{KZ}(\Delta(\nu)) = \dim(\nu)$ and thus $\dim \text{End}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{\text{KZ}}) = |W|$. \square

Let \mathcal{O}_{tor} be the Serre subcategory of \mathcal{O} consisting of all modules that are torsion with respect to the Ore set $\{\delta^N\}_{N \in \mathbb{N}}$. The torsion submodule M_{tor} of $M \in \mathcal{O}$ is the set $\{m \in M \mid \exists N \gg 0 \text{ s.t. } \delta^N \cdot m = 0\}$. Then, M is torsion if $M_{\text{tor}} = M$.

Corollary 4.7. *The KZ-functor is a quotient functor with kernel \mathcal{O}_{tor} i.e.*

$$\text{KZ} : \mathcal{O}/\mathcal{O}_{\text{tor}} \xrightarrow{\sim} \mathcal{H}_{\mathbf{q}}(W)\text{-mod}.$$

Proof. Notice that, of all the functors in diagram 11, only the first, $M \mapsto M[\delta^{-1}]$ is not an equivalence. We have $M[\delta^{-1}] = 0$ if and only if M is torsion. Therefore, $\mathbf{KZ}(M) = 0$ if and only if $M \in \mathcal{O}_{\text{tor}}$. Thus, we just need to show that \mathbf{KZ} is essentially surjective. Let N be a finite dimensional $\mathcal{H}_{\mathbf{q}}(W)$ -module. Recall that $P_{KZ} \in \mathcal{O}$ is a $(\mathbf{H}_{\mathbf{c}}(W), \mathcal{H}_{\mathbf{q}}(W))$ -bimodule. Therefore, $\text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, \mathbf{H}_{\mathbf{c}}(W))$ is a $(\mathcal{H}_{\mathbf{q}}(W), \mathbf{H}_{\mathbf{c}}(W))$ -bimodule and

$$M = \text{Hom}_{\mathcal{H}_{\mathbf{q}}(W)}(\text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, \mathbf{H}_{\mathbf{c}}(W)), N)$$

is a module in category \mathcal{O} . Applying (14) and the double centralizer theorem, Theorem 4.6, we have

$$\begin{aligned} \mathbf{KZ}(M) &= \text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, M) \\ &= \text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, \text{Hom}_{\mathcal{H}_{\mathbf{q}}(W)}(\text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, \mathbf{H}_{\mathbf{c}}(W)), N)) \\ &\simeq \text{Hom}_{\mathcal{H}_{\mathbf{q}}(W)}(\text{Hom}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}, \mathbf{H}_{\mathbf{c}}(W)) \otimes_{\mathbf{H}_{\mathbf{c}}(W)} P_{KZ}, N) \\ &\simeq \text{Hom}_{\mathcal{H}_{\mathbf{q}}(W)}(\text{End}_{\mathbf{H}_{\mathbf{c}}(W)}(P_{KZ}), N) \\ &\simeq \text{Hom}_{\mathcal{H}_{\mathbf{q}}(W)}(\mathcal{H}_{\mathbf{q}}(W), N) = N, \end{aligned}$$

where we have used [2, Proposition 4.4 (b)] in the third line. \square

4.7. Example. Let's take $W = \mathbb{Z}_n$. In this case the Hecke algebra $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_n)$ is generated by a single element $T := T_1$ and satisfies the defining relation

$$\prod_i (T - q_i^{m_i}) = 0.$$

Unlike examples of higher rank, the algebra $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_n)$ is commutative. Let $\zeta \in \mathbb{C}^\times$ be defined by $s(x) = \zeta x$. We fix $\alpha_s = \sqrt{2}x$ and $\alpha_s^\vee = \sqrt{2}y$, which implies that $\lambda_s = \zeta$. Let $\Delta(i) = \mathbb{C}[x] \otimes e_i$ be the standard module associated to the simple \mathbb{Z}_n -module e_i , where $s \cdot e_i = \zeta^i e_i$. The module $\Delta(i)$ is free as a $\mathbb{C}[x]$ -module and the action of y is uniquely defined by $y \cdot (1 \otimes e_i) = 0$. Since

$$y = \partial_x - \sum_{i=1}^{m-1} \frac{2\mathbf{c}_i}{1 - \zeta^i} \frac{1}{x} (1 - s^i)$$

under the Dunkl embedding, we have $\Delta(i)[\delta^{-1}] = \mathbb{C}[x, x^{-1}] \otimes e_i$ with connection defined by

$$\partial_x \cdot e_i = \frac{a_i}{x} e_i$$

where

$$a_i := 2 \sum_{j=1}^{m-1} \frac{\mathbf{c}_j (1 - \zeta^{ij})}{1 - \zeta^j}.$$

It is clear that this connection is regular. Then, the horizontal sections sheaf of $\Delta(i)[\delta^{-1}]$ on $\mathfrak{h}_{\text{reg}}^{an}$ is dual to the sheaf of multivalued solutions $\mathbb{C} \cdot x^{a_i}$ of the differential equation $x \partial_x - a_i = 0$. But this is not what we want. Recall that $(\Delta(i)[\delta^{-1}])^{\mathbb{Z}_n}$ is a module for $\mathcal{D}(\mathfrak{h}_{\text{reg}})^{\mathbb{Z}_n} = \mathcal{D}(\mathfrak{h}_{\text{reg}}/\mathbb{Z}_n)$ and $\mathbf{KZ}(\Delta(i))$ is defined to be the horizontal sections of $(\Delta(i)[\delta^{-1}])^{\mathbb{Z}_n}$. Let $z = x^n$ so that $\mathbb{C}[\mathfrak{h}_{\text{reg}}/\mathbb{Z}_n] = \mathbb{C}[z, z^{-1}]$.

Then, an easy calculation shows that $\partial_z = \frac{1}{nx^{n-1}}\partial_x$ (check this!). Since

$$(\Delta(i)[\delta^{-1}])^{\mathbb{Z}_n} = \mathbb{C}[z, z^{-1}] \cdot (x^{n-i} \otimes e_i) =: \mathbb{C}[z, z^{-1}] \cdot u_i,$$

we see that

$$\partial_z \cdot (x^{n-i} \otimes e_i) = \frac{1}{nx^{n-1}}\partial_x \cdot (x^{n-i} \otimes e_i) = \frac{n-i+a_i}{nz}u_i.$$

Hence $\text{KZ}(\Delta(i))$ is dual to the local system of solutions $\mathbb{C} \cdot z^{b_i}$, where

$$b_i = \frac{n-i+a_i}{n}.$$

At this level, duality simply means considering instead the local system $\mathbb{C} \cdot z^{-b_i}$. The generator T of $\pi_1(\mathfrak{h}_{\text{reg}}/\mathbb{Z}_n)$ is represented by the loop $t \mapsto \exp(2\pi\sqrt{-1}t)$. Therefore

$$T \cdot z^{-b_i} = \exp(-2\pi\sqrt{-1}b_i)z^{-b_i}.$$

It turns out that, in the rank one case, $L(i)[\delta^{-1}] = 0$ if $L(i) \neq \Delta(i)$. Thus,

$$\text{KZ}(L(i)) = \begin{cases} \text{KZ}(\Delta(i)) & \text{if } L(i) = \Delta(i) \\ 0 & \text{otherwise.} \end{cases}$$

4.8. Application. As an application of the Double centralizer theorem, Theorem 4.6, we mention the following very useful result due to Vale, [50].

Theorem 4.8. *Assume that¹¹ $\dim \mathcal{H}_{\mathbf{q}}(W) = |W|$. Then the following are equivalent:*

- (1) $\mathbf{H}_{\mathbf{k}}(W)$ is a simple ring.
- (2) Category \mathcal{O} is semi-simple.
- (3) The cyclotomic Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$ is semi-simple.

4.9. The KZ functor for \mathbb{Z}_2 . In this section, which is designed to be an exercise for the reader, we'll try to describe what the KZ functor does to modules in category \mathcal{O} when $W = \mathbb{Z}_2$, our favourite example. The Hecke algebra $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_2)$ is the algebra generated by $T := T_1$ and satisfying the relation $(T-1)(T-\mathbf{q}) = 0$. The defining relation for $\mathbf{H}_{\mathbf{c}}(\mathbb{Z}_2)$ is

$$[y, x] = 1 - 2\mathbf{c}s,$$

see example 1.11. We have $\mathbf{q} = \det(s) \exp(2\pi\sqrt{-1}\mathbf{c}) = -\exp(2\pi\sqrt{-1}\mathbf{c})$.

Exercise 4.9. Describe $\text{KZ}(\Delta(\lambda))$ as a $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_2)$ -module.

For the next exercise, you'll need the explicit description of $P(e_1)$ given in exercise 2.27.

Exercise 4.10. (Harder) Assuming $\mathbf{c} = \frac{1}{2} + m$ for some $m \in \mathbb{Z}_{\geq 0}$. Calculate $\text{KZ}(P(e_1))$.

Exercise 4.11. For all \mathbf{c} , describe P_{KZ} .

4.10. Additional remark.

- Most of the results in this lecture first appeared in [27] and our exposition is based mainly on this paper.

¹¹The equality $\dim \mathcal{H}_{\mathbf{q}}(W) = |W|$ is known to hold for all but finitely many exceptional complex reflection groups. It is conjectured to always hold.

- Further details on the KZ-functor are also contained in [44].

5. SYMPLECTIC REFLECTION ALGEBRAS AT $t = 0$

Recall from the first lecture that we used the Satake isomorphism to show that

- The algebra $Z(\mathbf{H}_{0,\mathbf{c}}(G))$ is isomorphic to $\mathbf{e}\mathbf{H}_{0,\mathbf{c}}(G)\mathbf{e}$ and $\mathbf{H}_{0,\mathbf{c}}(G)$ is a finite $Z(\mathbf{H}_{0,\mathbf{c}}(G))$ -module.
- The centre of $\mathbf{H}_{1,\mathbf{c}}(G)$ equals \mathbb{C} .

In this lecture we'll consider symplectic reflection algebras “at $t = 0$ ” and, in particular, the geometry of $Z_{\mathbf{c}}(G) := Z(\mathbf{H}_{0,\mathbf{c}}(G))$.

Definition 5.1. The *generalized Calogero-Moser space* $X_{\mathbf{c}}(G)$ is defined to be the affine variety $\text{Spec } Z_{\mathbf{c}}(G)$.

The filtration on $\mathbf{H}_{0,\mathbf{c}}(G)$ induces, by restriction, a filtration on $Z_{\mathbf{c}}(G)$. Since the associated graded of $Z_{\mathbf{c}}(G)$ is $\mathbb{C}[V]^G$, $X_{\mathbf{c}}(G)$ is reduced and irreducible.

Example 5.2. When $G = \mathbb{Z}_2$ acts on \mathbb{C}^2 , the centre of $\mathbf{H}_{\mathbf{c}}(\mathbb{Z}_2)$ is generated by $A := x^2, B := xy - \mathbf{c}s$ and $C = y^2$. Thus,

$$X_{\mathbf{c}}(\mathbb{Z}_2) \simeq \frac{\mathbb{C}[A, B, C]}{(AC - (B + \mathbf{c})(B - \mathbf{c}))}$$

is the affine cone over $\mathbb{P}^1 \subset \mathbb{P}^2$ when $\mathbf{c} = 0$, but is a smooth affine surface for all $\mathbf{c} \neq 0$, see figure 1.1.

5.1. Representation Theory. Much of the geometry of the generalized Calogero-Moser space is encoded in the representation theory of the corresponding symplectic reflection algebra (a consequence of the double centralizer property!). In particular, a closed point of $X_{\mathbf{c}}$ is singular if and only if there is a “small” simple module supported at that point - this statement is made precise in Proposition 5.6 below.

Lemma 5.3. *Let L be a simple $\mathbf{H}_{0,\mathbf{c}}(G)$ -module. Then $\dim L < \infty$.*

Proof. It is a consequence of Kaplansky's Theorem, [41, Theorem 13.3.8], that every simple $\mathbf{H}_{0,\mathbf{c}}(G)$ -module is a finite dimensional vector space over \mathbb{C} . More precisely, if L is a simple $\mathbf{H}_{0,\mathbf{c}}(G)$ -module then $\dim L \leq \text{P.I. degree}(\mathbf{H}_{0,\mathbf{c}}(G))$ and $\mathbf{H}_{0,\mathbf{c}}(G)/\text{Ann}_{\mathbf{H}_{0,\mathbf{c}}(G)} L \simeq \text{Mat}_m(\mathbb{C})$. \square

Schur's lemma says that the elements of the centre $Z_{\mathbf{c}}(G)$ of $\mathbf{H}_{0,\mathbf{c}}(G)$ act as scalars on any simple $\mathbf{H}_{0,\mathbf{c}}(G)$ -module L . Therefore, the simple module L defines a character $\chi_L : Z_{\mathbf{c}}(G) \rightarrow \mathbb{C}$ and the kernel of χ_L is a maximal ideal in $Z_{\mathbf{c}}(G)$. Thus, the character χ_L corresponds to a closed point in $X_{\mathbf{c}}(G)$. Without loss of generality, we will refer to this point as χ_L and denote by $Z_{\mathbf{c}}(G)_{\chi_L}$ the localization of $Z_{\mathbf{c}}(G)$ at the maximal ideal $\text{Ker } \chi_L$. We denote by $\mathbf{H}_{0,\mathbf{c}}(G)_{\chi}$ the central localization $\mathbf{H}_{0,\mathbf{c}}(G) \otimes_{Z_{\mathbf{c}}(G)} Z_{\mathbf{c}}(G)_{\chi}$. The Azumaya locus of $\mathbf{H}_{0,\mathbf{c}}(G)$ over $Z_{\mathbf{c}}(G)$ is defined to be

$$\mathcal{A}_{\mathbf{c}} := \{\chi \in X_{\mathbf{c}}(W) \mid \mathbf{H}_{0,\mathbf{c}}(W)_{\chi} \text{ is Azumaya over } Z_{\mathbf{c}}(W)_{\chi}\}.$$

As shown in [11, Theorem III.1.7], $\mathcal{A}_{\mathbf{c}}$ is a non-empty, open subset of $X_{\mathbf{c}}(W)$.

Remark 5.4. If you are not familiar with the (slightly technical) definition of Azumaya locus, as given in [10, Section 3], then it suffices to note that it is a consequence of the Artin-Procesi Theorem [41, Theorem 13.7.14] that the following are equivalent:

- (1) $\chi \in \mathcal{A}_{\mathbf{c}}$;
- (2) $\dim L = \text{P.I. degree}(\mathbf{H}_{0,\mathbf{c}}(G))$ for all simple modules L such that $\chi_L = \chi$;
- (3) there exists a unique simple module L such that $\chi_L = \chi$.

In fact, one can say a great deal more about these simple modules of maximal dimension. The following result strengthens Lemma 5.3.

Theorem 5.5. *Let L be a simple $\mathbf{H}_{0,\mathbf{c}}(G)$ -module. Then $\dim L \leq |G|$ and $\dim L = |G|$ implies that $L \simeq \mathbb{C}G$ as a G -module.*

Corollary 5.6. *Let L be a simple $\mathbf{H}_{\mathbf{c}}(G)$ -module then $\dim L = |G|$ if and only if χ_L is a nonsingular point of $X_{\mathbf{c}}(G)$.*

Proof. By Theorem 5.5, the dimension of a generic simple module is $|W|$. Since the Azumaya locus $\mathcal{A}_{\mathbf{c}}$ is dense in $X_{\mathbf{c}}$, it follows that $\text{P.I. degree}(\mathbf{H}_{0,\mathbf{c}}(G)) = |G|$. The proposition will then follow from the equality $\mathcal{A}_{\mathbf{c}} = (X_{\mathbf{c}}(G))_{sm}$, where $(X_{\mathbf{c}}(G))_{sm}$ is the smooth locus of $X_{\mathbf{c}}(G)$. As noted in Corollary 2.22, $\mathbf{H}_{0,\mathbf{c}}(G)$ has finite global dimension. It is known, [11, Lemma III.1.8], that this implies that $\mathcal{A}_{\mathbf{c}} \subseteq (X_{\mathbf{c}})_{sm}$. The opposite inclusion is an application of a result by Brown and Goodearl, [10, Theorem 3.8]. Their theorem says that $(X_{\mathbf{c}})_{sm} \subseteq \mathcal{A}_{\mathbf{c}}$ (in fact that we have equality) if $\mathbf{H}_{0,\mathbf{c}}(G)$ has particularly nice homological properties - it must be Auslander-regular and Cohen-Macaulay, and the complement of $\mathcal{A}_{\mathbf{c}}$ has codimension at least two in $X_{\mathbf{c}}$. The fact that $\mathbf{H}_{0,\mathbf{c}}(G)$ is Auslander-regular and Cohen-Macaulay can be deduced from the fact that its associated graded, the skew group ring, has these properties (the results that are required to show this are listed in the proof of [9, Theorem 4.4]). The fact that the complement of $\mathcal{A}_{\mathbf{c}}$ has co-dimension at least two in $X_{\mathbf{c}}$ is harder to show. It follows from the fact that $X_{\mathbf{c}}$ is a symplectic variety, Theorem 5.13, and that the “representation theory of $\mathbf{H}_{0,\mathbf{c}}$ is constant along orbits”, Theorem 5.14. \square

The corollary implies that to answer the question

Question 5.7. Is the generalized Calogero-Moser space smooth?

it suffices to compute the dimension of simple $\mathbf{H}_{\mathbf{c}}(G)$ -modules. Unfortunately, this turns out to be rather difficult to do.

5.2. Poisson algebras. The extra parameter t gives us a canonical quantization of the space $X_{\mathbf{c}}(G)$. As a consequence, this implies that $X_{\mathbf{c}}(G)$ is a Poisson variety. Recall:

Definition 5.8. A *Poisson algebra* $(A, \{-, -\})$ is a commutative algebra with a bracket $\{-, -\} : A \otimes A \rightarrow A$ such that

- (1) The pair $(A, \{-, -\})$ is a Lie algebra.
- (2) $\{a, -\} : A \rightarrow A$ is a derivation for all $a \in A$ i.e.

$$\{a, bc\} = \{a, b\}c + b\{a, c\}, \quad \forall a, b, c \in A.$$

Hayashi’s construction, [34]: We may think of t as a variable so that $\mathbf{H}_{0,\mathbf{c}}(G) = \mathbf{H}_{t,\mathbf{c}}(G)/t \cdot \mathbf{H}_{t,\mathbf{c}}(G)$. For $z_1, z_2 \in \mathbf{Z}_{\mathbf{c}}(G)$ define

$$\{z_1, z_2\} = \left(\frac{1}{t} [\hat{z}_1, \hat{z}_2] \right) \bmod t\mathbf{H}_{t,\mathbf{c}}(G),$$

where \hat{z}_1, \hat{z}_2 are arbitrary lifts of z_1, z_2 in $H_{t,c}(G)$.

Proposition 5.9. *Since $H_{0,c}(W)$ is flat over $\mathbb{C}[t]$, $\{-, -\}$ is a well-defined Poisson bracket on $Z_c(G)$.*

Exercise 5.10. (1) Prove Proposition 5.9.

(2) Show that same construction makes $eH_{0,c}(G)e$ into a Poisson algebra such that the Satake isomorphism is an isomorphism of Poisson algebras.

5.3. Symplectic leaves. In the algebraic world there are several different definitions of symplectic leaves, which can be shown to agree in “good” cases. We will recall two of them here. First, assume that $X_c(G)$ is smooth. Then $X_c(G)$ may be considered as a complex analytic manifold equipped with the analytic topology. In this case, the *symplectic leaf* through $\mathfrak{m} \in X_c(G)$ is the maximal connected analytic submanifold $\mathcal{L}(\mathfrak{m})$ of $X_c(G)$ which contains \mathfrak{m} and on which $\{-, -\}$ is non-degenerate. An equivalent definition is to say that $\mathcal{L}(\mathfrak{m})$ is the set of all points that can be reached from \mathfrak{m} by traveling along integral curves corresponding to the Hamiltonian vector fields $\{z, -\}$ for $z \in Z_c(G)$.

Let us explain in more detail what is meant by this. Let v be a vector field on a complex analytic manifold X i.e. v is a holomorphic map $X \rightarrow TX$ such that $v(x) \in T_x X$ for all $x \in X$ (v is assigning, in a continuous manner, a tangent vector to each point of x). An integral curve for v through x is a holomorphic function $\Phi_{x,v} : B_\epsilon(0) \rightarrow X$, where $B_\epsilon(0)$ is a closed ball of radius ϵ around 0 in \mathbb{C} , such that $(d_0 \Phi_{x,v})(1) = v(x)$ i.e. the derivative of $\Phi_{x,v}$ at 0 maps the basis element 1 of $T_0 \mathbb{C} = \mathbb{C}$ to the tangent vector field $v(x)$. The existence and uniqueness of holomorphic solutions to ordinary differential equations implies that $\Phi_{x,v}$ exists, and is unique, for each choice of v and x .

Now assume that $v = \{a, -\}$ is a Hamiltonian vector field, and fix $x \in X$. Then, the image of $\Phi_{x,\{a,-\}}$ is, by definition, contained in the symplectic leaf \mathcal{L}_x through x . Picking another point $y \in \Phi_{x,\{a,-\}}(B_\epsilon(0))$ and another Hamiltonian vector field $\{b, -\}$, we again calculate the integral curve $\Phi_{y,\{b,-\}}$ and its image is again, by definition, contained in \mathcal{L}_x . Continuing in this way for as long as possible, \mathcal{L}_x is the set of all points one can reach from x by “flowing along Hamiltonian vector fields”.

In particular, this defines a stratification of $X_c(G)$. If, on the other hand, $X_c(G)$ is not smooth, then we first stratify the smooth locus of $X_c(G)$. The singular locus $X_c(G)_{\text{sing}}$ of $X_c(G)$ is a Poisson subvariety. Therefore, the smooth locus of $X_c(G)_{\text{sing}}$ is again a Poisson manifold and has a stratification by symplectic leaves. We can continue by considering the “the singular locus of the singular locus” of $X_c(G)$ and repeating the argument... This way we get a stratification of the whole of $X_c(G)$ by symplectic leaves.

5.4. Symplectic cores. Let \mathfrak{p} be a prime ideal in $Z_c(G)$. Then there is a (necessarily unique) largest Poisson ideal $\mathcal{P}(\mathfrak{p})$ contained in \mathfrak{p} . Define an equivalence relation \sim on $X_c(G)$ by saying

$$\mathfrak{p} \sim \mathfrak{q} \Leftrightarrow \mathcal{P}(\mathfrak{p}) = \mathcal{P}(\mathfrak{q}).$$

The *symplectic cores* of $X_c(G)$ are the equivalence classes defined by \sim . We write

$$\mathcal{C}(\mathfrak{p}) = \{\mathfrak{q} \in X_c(G) \mid \mathcal{P}(\mathfrak{p}) = \mathcal{P}(\mathfrak{q})\}.$$

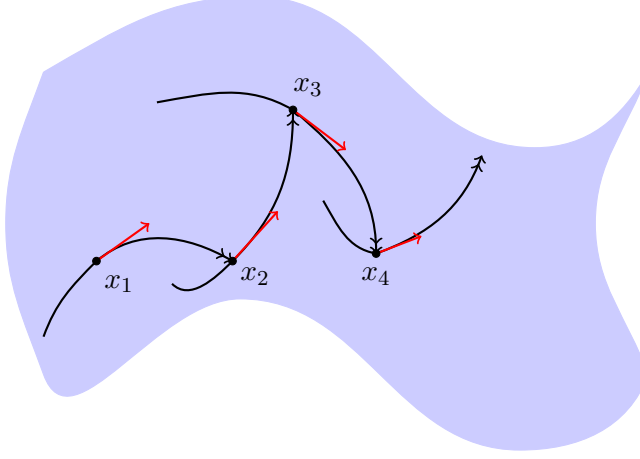


FIGURE 5. Flowing along integral curves in a symplectic leaf. The red vector at x_i is a Hamiltonian vector field and the curve through x_i is the corresponding integral curve.

Then, each symplectic core $\mathcal{C}(\mathfrak{p})$ is a locally closed subvariety of $X_{\mathbf{c}}(G)$ and $\overline{\mathcal{C}(\mathfrak{p})} = V(\mathcal{P}(\mathfrak{p}))$. The set of all symplectic cores is a partition of $X_{\mathbf{c}}(G)$ into locally closed subvarieties. As one can see from the examples below, a Poisson variety X will typically have an infinite number of symplectic leaves and an infinite number of symplectic cores.

Definition 5.11. We say that the Poisson bracket on X is *algebraic* if X has only finitely many symplectic leaves.

In this case it is known, [12, Proposition 3.7], that the symplectic leaves are locally closed algebraic sets and that the stratification by symplectic leaves equals the stratification by symplectic cores i.e.

$$\mathcal{L}(\mathfrak{m}) = \mathcal{C}(\mathfrak{m})$$

for all maximal ideals $\mathfrak{m} \in X_{\mathbf{c}}(G)$.

Exercise 5.12. (1) Consider the Poisson bracket on $\mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$ given by $\{x, y\} = y$.

Describe the symplectic leaves in \mathbb{C}^2 . Are they algebraic?

- (2) For each finite dimensional Lie algebra \mathfrak{g} , there is a natural Poisson bracket on $\mathbb{C}[\mathfrak{g}^*] = \text{Sym}(\mathfrak{g})$, uniquely defined by $\{X, Y\} = [X, Y]$ for all $X, Y \in \mathfrak{g}$. Recall that $\mathfrak{sl}_2 = \mathbb{C}\{E, F, H\}$ with $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$. Calculate the symplectic leaves of \mathfrak{sl}_2^* . The nullcone \mathcal{N} is defined to be $V(EF + \frac{1}{4}H^2)$. How many leaves are there in \mathcal{N} ? Hints: Calculate the integral curves of X_E, X_F and X_H through a point $(p, q, r) \in \mathfrak{sl}_2^*$. Show that each variety $V(EF + \frac{1}{4}H^2 = s)$, for $s \in \mathbb{C}$, is a Poisson subvariety.

In the case of symplectic reflection algebras, we have:

Theorem 5.13. *The symplectic leaves of the Poisson variety $X_{\mathbf{c}}(W)$ are precisely the symplectic cores of $X_{\mathbf{c}}(W)$. In particular, they are finite in number, hence the bracket $\{-, -\}$ is algebraic.*

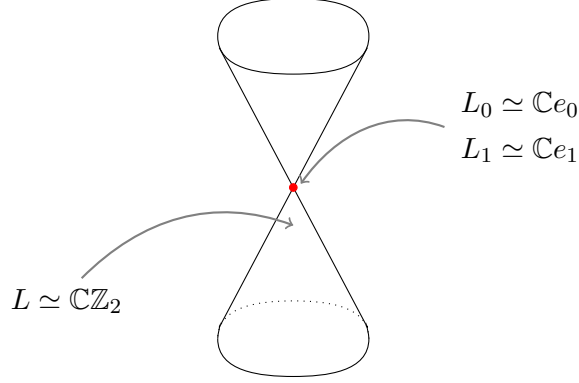


FIGURE 6. The leaves and simple modules of $H_{0,0}(\mathbb{Z}_2)$.

Remarkably, the representation theory of symplectic reflection algebras is “constant along symplectic leaves”, in the following precise sense. For each $\chi \in X_{\mathbf{c}}(G)$, let $H_{\chi,\mathbf{c}}(G)$ be the finite-dimensional quotient $H_{0,\mathbf{c}}(G)/\mathfrak{m}_{\chi}H_{0,\mathbf{c}}(G)$, where \mathfrak{m}_{χ} is the kernel of χ . If $\chi \in \mathcal{A}_{\mathbf{c}} = X_{\mathbf{c}}(G)_{\text{sm}}$ then

$$H_{\chi,\mathbf{c}}(G) \simeq \text{Mat}_{|G|}(\mathbb{C}), \quad \dim H_{\chi,\mathbf{c}}(G) = |G|^2.$$

This is not true if $\chi \in X_{\mathbf{c}}(G)_{\text{sing}}$.

Theorem 5.14. *Let χ_1, χ_2 be two points in \mathcal{L} , a symplectic leaf of $X_{\mathbf{c}}(G)$. Then,*

$$H_{\chi_1,\mathbf{c}}(G) \simeq H_{\chi_2,\mathbf{c}}(G).$$

Example 5.15. Let’s consider again our favorite example $W = \mathbb{Z}_2$. When \mathbf{c} is non-zero, one can check, as in the right hand side of figure 1.1, that $X_{\mathbf{c}}(\mathbb{Z}_2)$ is smooth. Therefore, it has only one symplectic leaf i.e. it is a symplectic manifold. Over each closed point of $X_{\mathbf{c}}(\mathbb{Z}_2)$ there is exactly one simple $H_{0,\mathbf{c}}(\mathbb{Z}_2)$ -module, which is isomorphic to $\mathbb{C}\mathbb{Z}_2$ as a \mathbb{Z}_2 -module. If, on the other hand, $\mathbf{c} = 0$ so that $H_{0,0}(\mathbb{Z}_2) = \mathbb{C}[x, y] \rtimes \mathbb{Z}_2$, then there is one singular point and hence two symplectic leaves - the singular point and its complement. On each closed point of the smooth locus, there is exactly one simple $\mathbb{C}[x, y] \rtimes \mathbb{Z}_2$ -module, which is again isomorphic to $\mathbb{C}\mathbb{Z}_2$ as a \mathbb{Z}_2 -module. However, above the singular point there are two simple, one-dimensional, modules, isomorphic to $\mathbb{C}e_0$ and $\mathbb{C}e_1$ as $\mathbb{C}\mathbb{Z}_2$ -modules. See figure 5.4.

5.5. Restricted rational Cherednik algebras. In order to be able to say more about the simple modules for $H_{0,\mathbf{c}}(G)$, e.g. to describe their possible dimensions, we restrict ourselves to considering rational Cherednik algebras. Therefore, in this subsection, we let W be a complex reflection group and $H_{0,\mathbf{c}}(W)$ the associated rational Cherednik algebra, as defined in lecture one. In the case of Coxeter groups the following was proved in [24, Proposition 4.15], and the general case is due to [29, Proposition 3.6].

Proposition 5.16. *Let $H_{0,\mathbf{c}}(W)$ be a rational Cherednik algebra associated to the complex reflection group W .*

- (1) *The subalgebra $\mathbb{C}[\hbar]^W \otimes \mathbb{C}[\hbar^*]^W$ of $H_{0,\mathbf{c}}(W)$ is contained in $Z_{\mathbf{c}}(W)$.*

(2) The centre $Z_{\mathbf{c}}(W)$ of $H_{0,\mathbf{c}}(W)$ is a free $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ -module of rank $|W|$.

The inclusion of algebras $A := \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{\mathbf{c}}(W)$ allows us to define the *restricted rational Cherednik algebra* $\overline{H}_{\mathbf{c}}(W)$ as

$$\overline{H}_{\mathbf{c}}(W) = \frac{H_{\mathbf{c}}(W)}{A_+ \cdot H_{\mathbf{c}}(W)},$$

where A_+ denotes the ideal in A of elements with zero constant term. This algebra was originally introduced, and extensively studied, in the paper [29]. The PBW theorem implies that

$$\overline{H}_{\mathbf{c}}(W) \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]^{coW}$$

as vector spaces. Here

$$\mathbb{C}[\mathfrak{h}]^{coW} = \mathbb{C}[\mathfrak{h}] / \langle \mathbb{C}[\mathfrak{h}]_+^W \rangle$$

is the *coinvariant algebra*. Since W is a complex reflection group, $\mathbb{C}[\mathfrak{h}]^{coW}$ has dimension $|W|$ and is isomorphic to the regular representation as a W -module. Thus, $\dim \overline{H}_{\mathbf{c}}(W) = |W|^3$. Denote by $\text{Irr}(W)$ a set of complete, non-isomorphic simple W -modules.

Definition 5.17. Let $\lambda \in \text{Irr}(W)$. The *baby Verma module* of $\overline{H}_{\mathbf{c}}(W)$, associated to λ , is

$$\Delta(\lambda) := \overline{H}_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \lambda,$$

where $\mathbb{C}[\mathfrak{h}^*]_+^{coW}$ acts on λ as zero.

We summarize, as is done in [29, Proposition 4.3] the results of [35] applied to this situation.

Proposition 5.18. Let $\lambda, \mu \in \text{Irr}(W)$.

- The baby Verma module $\Delta(\lambda)$ has a simple head, $L(\lambda)$. Hence $\Delta(\lambda)$ is indecomposable.
- $\Delta(\lambda)$ is isomorphic to $\Delta(\mu)$ if and only if $\lambda \simeq \mu$.
- The set $\{L(\lambda) \mid \lambda \in \text{Irr}(W)\}$ is a complete set of pairwise non-isomorphic simple $\overline{H}_{\mathbf{c}}(W)$ -modules.

Both the rational Cherednik algebra and the restricted rational Cherednik algebra are \mathbb{Z} -graded (no such grading exists for general symplectic reflection algebras). This means that the representation theory of $\overline{H}_{\mathbf{c}}(W)$ has a rich combinatorial structure and one can use some of the combinatorics to better describe the modules $L(\lambda)$.

5.6. The Calogero-Moser partition. Since the algebra $\overline{H}_{\mathbf{c}}(W)$ is finite dimensional, it will decompose into a direct sum of *blocks*:

$$\overline{H}_{\mathbf{c}}(W) = \bigoplus_{i=1}^k B_i,$$

where B_i is a block if it is indecomposable as an algebra. If b_i is the identity element of B_i then the identity element 1 of $\overline{H}_{\mathbf{c}}(W)$ is the sum $1 = b_1 + \cdots + b_k$ of the b_i . For each simple $\overline{H}_{\mathbf{c}}(W)$ -module L , there exists a unique i such that $b_i \cdot L \neq 0$. In this case we say that L belongs to the block B_i . By Proposition 5.18, we can (and will) identify $\text{Irr } \overline{H}_{\mathbf{c}}(W)$ with $\text{Irr}(W)$. Following [32] we define the *Calogero-Moser partition* of $\text{Irr}(W)$ to be the set of equivalence classes of $\text{Irr}(W)$ under the

equivalence relation $\lambda \sim \mu$ if and only if $L(\lambda)$ and $L(\mu)$ belong to the same block.

To aid intuition it is a good idea to have a geometric interpretation of the Calogero-Moser partition. It is a consequence of a theorem by Müller, see [13, Corollary 2.7], that the primitive central idempotents of $\overline{H}_c(W)$ (the b_i 's) are the images of the primitive idempotents of $Z_c/A_+ \cdot Z_c$ under the natural map $Z_c/A_+ \cdot Z_c \rightarrow \overline{H}_c(W)$. The inclusion $A \hookrightarrow Z_c(W)$ defines a finite, surjective morphism

$$\Upsilon : X_c(G) \longrightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W$$

where $\mathfrak{h}/W \times \mathfrak{h}^*/W = \text{Spec } A$. Geometrically, Müller's theorem is saying that the natural map $\text{Irr}(W) \rightarrow \Upsilon^{-1}(0)$, $\lambda \mapsto \text{Supp}(L(\lambda)) = \chi_{L(\lambda)}$, factors through the Calogero-Moser partition (here $\Upsilon^{-1}(0)$ is considered as the set theoretic pull-back):

$$\begin{array}{ccc} \text{Irr}(W) & & \\ \downarrow & \searrow & \\ CM(W) & \xrightarrow{\sim} & \Upsilon^{-1}(0) \end{array}$$

Using this fact, one can show that the geometry of $X_c(W)$ is related to Calogero-Moser partitions in the following way.

Theorem 5.19. *The following are equivalent:*

- *The generalized Calogero-Moser space $X_c(W)$ is smooth.*
- *The Calogero-Moser partition of $\text{Irr}(W')$ is trivial for all parabolic subgroup W' of W .*

Here W' is a *parabolic* subgroup of W if there exists some $v \in \mathfrak{h}$ such that $W' = \text{Stab}_W(v)$. It is a remarkable theorem by Steinberg that all parabolic subgroups of W are again complex reflection groups.

5.7. Symplectic resolutions. Now we return to the original question posed at the start of lecture one.

Definition 5.20. A *(projective) resolution of singularities* is a birational morphism $\pi : Y \rightarrow V/G$ from a smooth variety Y , projective over V/G , such that the restriction of π to $\pi^{-1}((V/G)_{\text{sm}})$ is an isomorphism.

If V_{reg} is the open subset of V on which G acts freely, then $V_{\text{reg}}/G \subset V/G$ is the smooth locus and it inherits a symplectic structure from V i.e. V_{reg}/G is a symplectic manifold.

Definition 5.21. A projective resolution of singularities $\pi : Y \rightarrow V/G$ is said to be *symplectic* if Y is a symplectic manifold and the restriction of π to $\pi^{-1}((V/G)_{\text{sm}})$ is an isomorphism of symplectic manifolds.

The existence of a symplectic resolution for V/G is a very strong condition and implies that the map π has some very good properties e.g. π is *semi-small*. Therefore, as one might expect, symplectic resolutions exist only for very special groups.

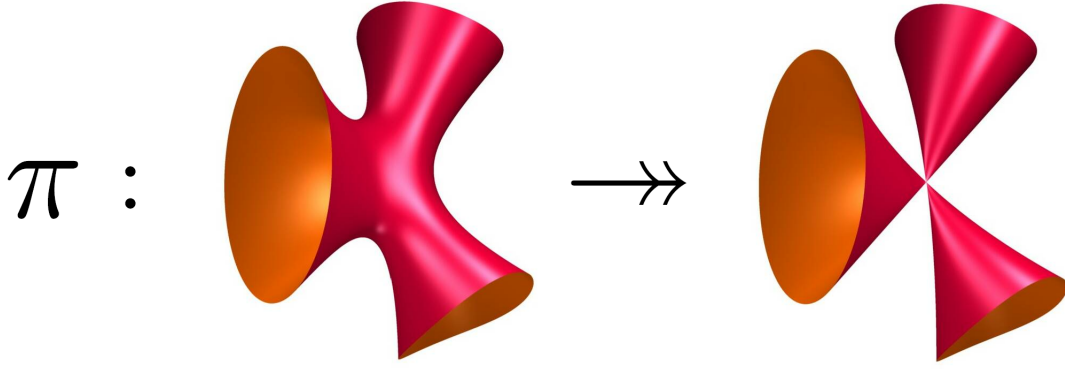


FIGURE 7. A representation of the resolution of the D_4 Kleinian singularity.

Theorem 5.22. *Let (V, ω, G) be an irreducible symplectic reflection group.*

- *The quotient singularity V/G admits a symplectic resolution if and only if it admits a smooth Poisson deformation.*
- *The quotient singularity V/G admits a smooth Poisson deformation if and only if $X_{\mathbf{c}}(G)$ is smooth for generic parameters \mathbf{c} .*

The irreducible symplectic reflection groups have been classified by Cohen, [16]. Using the above theorem, work of several people (Verbitsky [52], Ginzburg-Kaledin [28], Gordon [29], Bellamy [3], Bellamy-Schedler [5]) means that the classification of quotient singularities admitting symplectic resolutions is (almost) complete.

Example 5.23. Let $G \subset SL_2(\mathbb{C})$ be a finite group. Since $\dim \mathbb{C}^2/G = 2$, there is a *minimal resolution* $\widetilde{\mathbb{C}^2}/G$ of \mathbb{C}^2/G through which all other resolutions factor. This resolution can be explicitly constructed as a series of blowups. Moreover, $\widetilde{\mathbb{C}^2}/G$ is a symplectic manifold and hence provides a symplectic resolution of \mathbb{C}^2/G . The corresponding generalized Calogero-Moser space $X_{\mathbf{c}}(G)$ is smooth for generic parameters \mathbf{c} . The corresponding symplectic reflection algebras are closely related to deformed preprojective algebras, [17].

5.8. Graded characters. In this section, which is an exercise for the reader, we assume that W is a complex reflection group. The idea will be to use the grading on $H_{0,\mathbf{c}}(W)$ to try and describe the simple modules $L(\lambda)$.

- Exercise 5.24.*
- (1) Check that $\deg(x) = 1$, $\deg(y) = -1$ and $\deg(w) = 0$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$ defines a \mathbb{Z} -grading on $H_{t,\mathbf{c}}(W)$. Why does this also make $\overline{H}_{\mathbf{c}}(W)$ a graded algebra?
 - (2) Let $\lambda \in \text{Irr}(W)$. Show that the baby Verma module $\Delta(\lambda)$ is graded and that the simple module $L(\lambda)$ is a graded quotient of $\Delta(\lambda)$.
 - (3) Using the results from this lecture, show that λ is in a block on its own if and only if $\dim L(\lambda) = |W|$ if and only if $L(\lambda) \simeq \mathbb{C}W$ as a W -module.

Recall that $\mathbb{C}[\mathfrak{h}]^{coW}$, the coinvariant ring of W , is defined to be the quotient $\mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^W \rangle$. It is a graded W -module. Therefore, we can define the *fake degree* of $\lambda \in \text{Irr}(W)$ to be the polynomial

$$f_\lambda(t) = \sum_{i \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda] t^i$$

where $\mathbb{C}[\mathfrak{h}]_i^{coW}$ is the part of $\mathbb{C}[\mathfrak{h}]^{coW}$ of degree i and $[\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda]$ is the multiplicity of λ in $\mathbb{C}[\mathfrak{h}]_i^{coW}$. For each $\lambda \in \text{Irr}(W)$, we define b_λ to be the degree of smallest monomial appearing in $f_\lambda(t)$ e.g. if $f_\lambda(t) = 2t^4 - t^6 + \dots$ then $b_\lambda = 4$. Given a finite dimensional, graded vector space M , the Poincaré polynomial of M is defined to be

$$P(M, t) = \sum_{i \in \mathbb{Z}} \dim M_i t^i.$$

Exercise 5.25. (Harder) Assuming that λ is in a block on its own, show that the Poincaré polynomial of $L(\lambda)$ is given by

$$P(L(\lambda), t) = \frac{\dim \lambda t^{b_\lambda} P(\mathbb{C}[\mathfrak{h}]^{coW}, t)}{f_{\lambda^*}(t)}.$$

Hints:

- (1) Recall that $\Delta(\lambda)$ is indecomposable. So which $L(\mu)$ can be composition factors of $\Delta(\lambda)$?
- (2) Try to calculate the graded multiplicities of $L(\lambda)$ in $\Delta(\lambda)$.
- (3) Note that the trivial representation occurs only once in $L(\lambda)$. How often does it occur in $\Delta(\lambda)$?
- (4) Recall that for $\lambda, \mu \in \text{Irr}(W)$, $[\mu \otimes \lambda : \text{triv}] \neq 0$ if and only if $\mu \simeq \lambda^*$.

The module $L(\lambda)$ is finite dimensional. Therefore $P(L(\lambda), t)$ is a Laurent polynomial. However, one can often find representations λ for which $f_{\lambda^*}(t)$ does not divide $P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$. In such cases the above calculation show that λ is *never* in a block on its own.

Exercise 5.26. The character table of the Weyl group G_2 is given by

Class	1	2	3	4	5	6
Size	1	1	3	3	2	2
Order	1	2	2	2	3	6
T	1	1	1	1	1	1
S	1	1	-1	-1	1	1
V_1	1	-1	1	-1	1	-1
V_2	1	-1	-1	1	1	-1
\mathfrak{h}_1	2	2	0	0	-1	-1
\mathfrak{h}_2	2	2	0	0	-1	1

The fake polynomials are

$$f_T(t) = 1, \quad f_S(t) = t^6, \quad f_{V_1}(t) = t^3, \quad f_{V_2}(t) = t^3, \quad f_{\mathfrak{h}_1}(t) = t^2 + t^4, \quad f_{\mathfrak{h}_2}(t) = t + t^5.$$

Is $X_{\mathbf{c}}(G_2)$ ever smooth?

Exercise 5.27. (Harder) For this exercise you'll need to have GAP 3, together with the package "CHEVIE" installed. Using the code¹² `fake.gap`, show that there is (at most one) exceptional complex reflection group W for which the space $X_{\mathbf{c}}(W)$ can ever hope to be smooth. Which exceptional group is this?

5.9. Additional remark.

- Theorem 5.5 is proven in [24, Theorem 1.7].
- The fact that $X_{\mathbf{c}}(W)$ has finitely many symplectic leaves, Theorem 5.13, is [12, Theorem 7.8].
- The beautiful result that the representation theory of symplectic reflection algebras is constant along leaves, Theorem 5.14, is due to Brown and Gordon, [12, Theorem 4.2].
- Theorem 5.19 is stated for rational Cherednik algebras at $t = 1$ in positive characteristic in [4, Theorem 1.3]. However, the proof given there applies word for word to rational Cherednik algebras at $t = 0$ in characteristic zero.
- Steinberg's Theorem that the stabilizer subgroup of a complex reflection group is itself a complex reflection group is given in [48, Theorem 1.5].
- Theorem 5.22 follows from the results in [28] and [42].

¹²Available from <http://www.maths.gla.ac.uk/~gbellamy/MSRI.html>.

6. SOLUTIONS TO EXERCISES

We include (partial) solutions to the exercises scattered throughout the course notes.

6.1. Solutions to exercises in lecture 1.

Solution to exercise 1.9. Let $z = \sum_{g \in G} f_g \cdot g \in Z(\mathbb{C}[V] \rtimes G)$. Choose some $g \neq 1$. Since $G \subset GL(V)$, there exists $h \in \mathbb{C}[V]$ such that $g(h) \neq h$. Then

$$[h, z] = \sum_{g \in G} f_g (h - g(h)) \cdot g = 0$$

implies that $f_g = 0$ for all $g \neq 1$. Therefore $Z(\mathbb{C}[V] \rtimes G) \subset \mathbb{C}[V]$. But it is clear that $Z(\mathbb{C}[V] \rtimes G) \cap \mathbb{C}[V] \subseteq \mathbb{C}[V]^G$. On the other hand one can easily see that $\mathbb{C}[V]^G \subset Z(\mathbb{C}[V] \rtimes G)$. \square

Solution to exercise 1.16. Taking away the relation $[y, x] = 1$ from $(y - s)x = xy + s$ gives $sx = -s$ and hence $x = 1$. Then $[y, x] = 1$ implies that $1 = 0$. \square

Solution to exercise 1.23. Part (1): This is a direct calculation using the fact that α_s divides $f - s(f)$ for all $f \in \mathbb{C}[\mathfrak{h}]$. Similar, part (2) is also a direct calculation. \square

Solution to exercise 1.24. It is clear from the defining relations that every element in $H_{t,c}(W)$ can be expressed as a linear combination of elements $f \cdot w \cdot g$, where $f \in \mathbb{C}[\mathfrak{h}]$ and $g \in \mathbb{C}[\mathfrak{h}^*]$ are monomials and $w \in W$. Therefore, we need to show that these elements are linearly independent. One can define an order filtration on $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ by placing $\mathbb{C}[\mathfrak{h}_{\text{reg}}] \rtimes W$ in degree zero and y 's in degree one. The PBW theorem for $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ is then the statement that the associated graded algebra with respect to this filtration is $\mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*] \times W$. One can define a similar filtration \mathcal{F}_\bullet on $H_{t,c}(W)$ by placing $\mathbb{C}[\mathfrak{h}] \rtimes W$ in degree zero and \mathfrak{h} in degree one. The Dunkl embedding is filtration preserving and if $h \in H_{t,c}(W)$ has degree d say then its image in $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W$ will have degree d too. This implies that the associated graded map is just the embedding $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W \hookrightarrow \mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*] \rtimes W$. In particular, the PBW theorem holds for $H_{t,c}(W)$. \square

Solution to exercise 1.25. At $t = 0$, $\mathcal{D}_t(\mathfrak{h}_{\text{reg}}) \rtimes W = \mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*] \rtimes W$. Therefore the image of $eH_{0,c}(W)e$ is contained in $e\mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*] \rtimes We \simeq \mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*]^W$. This is a commutative ring. \square

Solution to exercise 1.26. Let $w \in W$ and $s \in \mathcal{S}$. Then $ws w^{-1} \in \mathcal{S}$ is again a reflection. This implies that there is some non-zero scalar β such that $w(\alpha_s) = \beta \alpha_{ws w^{-1}}$. Thus, $w(\delta) = \gamma \delta$ for some non-zero scalar γ . This implies that δ is a semi-invariant. Now the claim that, after localizing at δ , the Dunkl embedding becomes an isomorphism is clear from the definition of Dunkl operators because it becomes possible to express the generators y in terms of the D_y . \square

Solution to exercise 1.27. Firstly, $eH_{1,c}(W)e$ is an integral domain because it is a filtered ring whose associated graded is $\mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*]^W$. Since $\mathbb{C}[\mathfrak{h}_{\text{reg}} \times \mathfrak{h}^*]^W$ is an integral domain, so too is $eH_{1,c}(W)e$. Now choose $z \in Z(eH_{1,c}(W)e)$. The Dunkl embedding defines an isomorphism

$$eH_{1,c}(W)e[(e\delta^r)^{-1}] \xrightarrow{\sim} e\mathcal{D}(\mathfrak{h}_{\text{reg}}) \rtimes We \simeq \mathcal{D}(\mathfrak{h}_{\text{reg}})^W.$$

Therefore the image of z in $eH_{1,c}(W)e[(e\delta^r)^{-1}]$ is either a unit or zero. If it is a unit then $\alpha(e\delta^r)^a \cdot z = 1$ in $eH_{1,c}(W)e$, for some $\alpha \in \mathbb{C}^\times$ and $a \in \mathbb{N}$. But $e\delta^r$ is not a unit in $eH_{1,c}(W)e$ (since the symbol

of $e\delta^r$ in $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ is not a unit). Therefore $a = 0$ and $z \in \mathbb{C}^\times$. On the other hand if $(e\delta^r)^a \cdot z = 0$ for some a then the fact that $eH_{1,\mathbf{c}}(W)e$ is a domain implies that $z = 0$. \square

Solution to exercise 1.28. Part (1): Since the isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$, $x \mapsto \tilde{x} = (x, -)$ is W -equivariant, the only thing to check is that the commutation relation

$$[y, x] = tx(y) - \sum_{s \in \mathcal{S}} \mathbf{c}(s) \alpha_s(y) x(\alpha_s^\vee) s, \quad \forall y \in \mathfrak{h}, x \in \mathfrak{h}^*$$

still holds after applying \sim everywhere. This follows from two trivial observations. First,

$$x(y) = \tilde{y}(\tilde{x}) = (x, \tilde{y}) = (y, \tilde{x}),$$

and secondly, possibly after some rescaling, we have $(\alpha_s, \alpha_s) = (\alpha_s^\vee, \alpha_s^\vee) = 2$ and $\tilde{\alpha}_s = \alpha_s^\vee$, $\tilde{\alpha}_s^\vee = \alpha_s$. For part (2), just follow the hint. \square

6.2. Solutions to exercises in lecture 2.

Solution to exercise 2.4. The claim is just a direct calculation using the defining relations of $H_{\mathbf{c}}(W)$. \square

Solution to exercise 2.6. Part (1): Choose some $0 \neq x \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$ and take $M = H_{\mathbf{c}}(W)/H_{\mathbf{c}}(W) \cdot (\mathbf{eu} - x)$. This module cannot be a direct sum of its generalized eigenspaces.

Part (2): Let L be a finite-dimensional $H_{\mathbf{c}}(W)$ -module. Then, it is a direct sum of its generalized \mathbf{eu} -eigenspaces because $\mathbf{eu} \in \text{End}_{\mathbb{C}}(L)$ and a finite dimensional vector space decomposes as a direct sum of generalized eigenspaces under the action of any linear operator. If $l \in L_a$ for some $a \in \mathbb{C}$, then the relation $[\mathbf{eu}, y] = -y$ implies that $y \cdot l \in L_{a-1}$. Hence $y_1 \cdots y_k \cdot l \in L_{a-k}$. But L is finite dimensional which implies that $L_{a-k} = 0$ for $k \gg 0$. Hence \mathfrak{h} acts locally nilpotently on L . \square

Solution to exercise 2.7. Since M is finitely generate as a $\mathbb{C}[\mathfrak{h}]$ -module, we may choose a finite dimensional, \mathbf{eu} and \mathfrak{h} -stable subspace M_0 of M such that M_0 generates M as a $\mathbb{C}[\mathfrak{h}]$ -module. Therefore, there is a surjective map of $\mathbb{C}[\mathfrak{h}]$ -modules $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} M_0 \rightarrow M$, $a \otimes m \mapsto am$. This can be made into a morphism of $\mathbb{C}[\mathbf{eu}]$ -modules by defining $\mathbf{eu} \cdot (a \otimes m) = [\mathbf{eu}, a] \otimes m + a \otimes \mathbf{eu} \cdot m$. If $f_0(t) \in \mathbb{Z}[x^a \mid a \in \mathbb{C}]$ is the character of M_0 then the character of $\mathbb{C}[\mathfrak{h}] \otimes M_0$ is $\frac{1}{(1-t)^n} f_0(t) \in \bigoplus_{a \in \mathbb{C}} t^a \mathbb{Z}[[t]]$ and hence $\text{ch}(M) \in \bigoplus_{a \in \mathbb{C}} t^a \mathbb{Z}[[t]]$ too. For the second part, we note that the character of M only depends on its image in $K_0(\mathcal{O})$. Therefore if

$$[M] = \sum_{\lambda \in \text{Irr}(W)} n_{\lambda} [\Delta(\lambda)] \in K_0(\mathcal{O}),$$

for some $n_{\lambda} \in \mathbb{Z}$, then the fact that $\text{ch}(\Delta(\lambda)) = \frac{\dim(\lambda)t^{\mathbf{c}\lambda}}{(1-t)^n}$ implies that $\text{ch}(M) = \frac{1}{(1-t)^n} \cdot f(t)$, where

$$f(t) = \sum_{\lambda \in \text{Irr}(W)} n_{\lambda} \dim(\lambda) t^{\mathbf{c}\lambda} \in \mathbb{Z}[x^a \mid a \in \mathbb{C}].$$

\square

Solution to exercise 2.14. Let $M \in \mathcal{O}$. We can decompose M as a $\mathbb{C}[\mathbf{eu}]$ -module as

$$M = \bigoplus_{\bar{a} \in \mathbb{C}/\mathbb{Z}} M^{\bar{a}}$$

where $M^{\bar{a}} = \bigoplus_{b \in \bar{a}} M_b$. It suffices to show that each $M^{\bar{a}}$ is a $H_{\mathbf{c}}(W)$ -submodule of M . But if $x \in \mathfrak{h}^*$ and $m \in M_b$ then $x \cdot m \in M_{b+1}$ and $b \in \bar{a}$ iff $b+1 \in \bar{a}$. A similar argument applies to $y \in \mathfrak{h}^*$ and $w \in W$. Thus $M^{\bar{a}}$ is a $H_{\mathbf{c}}(W)$ -submodule of M . \square

Solution to exercise 2.15. Note that $\Delta(\lambda)_{\mathbf{c}_\lambda} = 1 \otimes \lambda$ and $\Delta(\lambda)_b \neq 0$ implies that $b - \mathbf{c}_\lambda \in \mathbb{Z}_{\geq 0}$. If the quotient map $\Delta(\lambda) \rightarrow L(\lambda)$ has a non-zero kernel K then choose $L(\mu) \subset K$ a simple submodule. We have $\mathbf{c}_\mu - \mathbf{c}_\lambda \in \mathbb{Z}_{>0}$, contradicting the minimality of k_0 . Thus $K = 0$. In a similar way, one can show that

$$\mathrm{Hom}_{\mathcal{O}^{\geq k_0}}(\Delta(\lambda), M) = \mathrm{Hom}_W(\lambda, M_{\mathbf{c}_\lambda})$$

for all $M \in \mathcal{O}^{\geq k_0}$. Since M is a direct sum of its generalized \mathbf{eu} -eigenspaces, this implies that $\mathrm{Hom}_{\mathcal{O}^{\geq k_0}}(\Delta(\lambda), -)$ is an exact functor i.e. $\Delta(\lambda)$ is projective. \square

Solution to exercise 2.20. Since \mathcal{O} is a finite length, abelian category with finitely many simple modules, those simple modules $L(\lambda)$ are a \mathbb{Z} -basis of $K_0(\mathcal{O})$. Let $k = |\mathrm{Irr}(W)|$ and define the k by k matrix $A = (a_{\lambda, \mu}) \in \mathbb{N}$ by

$$[\Delta(\lambda)] = \sum \mu \in \mathrm{Irr}(W) a_{\lambda, \mu} [L(\mu)].$$

We order $\mathrm{Irr}(W) = \{\lambda_1, \dots, \lambda_k\}$ so that $i < j$ implies that $\lambda_i \geq \lambda_j$. Then property (2) of definition 2.17 implies that A is upper triangular with ones all along the diagonal. This implies that A is invertible over \mathbb{Z} and hence the $[\Delta(\lambda)]$ are a basis of $K_0(\mathcal{O})$. \square

Solution to exercise 2.23. As in the proof of Theorem 3.11, it suffices to consider modules in the block $\mathcal{O}^{\bar{a}}$ for some $a \in \mathbb{C}$. Recall that we constructed a filtration of this category

$$\mathcal{O}^{\geq k_0} \subset \mathcal{O}^{\geq k_0 - i_1} \subset \dots \subset \mathcal{O}^{\geq k_0 - i_n} = \mathcal{O}^{\bar{a}},$$

where $0 < i_1 < \dots < i_n$ are chosen so that each inclusion $\mathcal{O}^{\geq k_0 - i_r} \subset \mathcal{O}^{\geq k_0 - i_{r+1}}$ is proper. For all $\lambda \in \mathrm{Irr}(W)$, define $N(\lambda)$ to be the positive integer such that $L(\lambda) \in \mathcal{O}^{\geq k_0 - i_{N(\lambda)}}$ but $L(\lambda) \notin \mathcal{O}^{\geq k_0 - i_{N(\lambda)-1}}$. We claim that $\mathrm{p.d.}(\Delta(\lambda)) \leq n - N(\lambda)$ for all λ . The proof of Theorem 2.13 showed that $\Delta(\lambda) = P(\lambda)$ for all λ such that $N(\lambda) = n$. Therefore we may assume that the claim is true for all μ such that $N(\mu) > N_0$. Choose λ such that $N(\lambda) = N_0$. Then, as shown in the proof of Theorem 3.18, we have a short exact sequence

$$0 \rightarrow K \rightarrow P(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0,$$

where K admits a filtration by $\Delta(\mu)$'s for $\lambda <_{\mathbf{c}} \mu$. The inductive hypothesis, together with standard homological results e.g. Chapter 4 of [53], imply that $\mathrm{p.d.}(K) \leq N_0 - 1$ and hence $\mathrm{p.d.}(\Delta(\lambda)) \leq N_0$ as required. Now we show, again by induction, that $\mathrm{p.d.}(L(\lambda)) \leq n + N(\lambda)$. It was shown in exercise 3.13 that $\Delta(\lambda) = L(\lambda)$ for all λ such that $N(\lambda) = 0$. Therefore, we may assume by induction that

the claim holds for all μ such that $N(\mu) < N_0$. Assume $N(\lambda) = N_0$. We have a short exact sequence

$$0 \rightarrow R \rightarrow \Delta(\lambda) \rightarrow L(\lambda) \rightarrow 0,$$

where R admits a filtration by simple modules $L(\mu)$ with $\mu <_{\mathbf{c}} \lambda$. Hence, as for standard modules, we may conclude that $\text{p.d.}(L(\lambda)) \leq n + N(\lambda)$. Note that we have actually shown that the global dimension of $\mathcal{O}^{\bar{a}}$ is at most $2n$. \square

Solution to exercise 2.25. Exercise 2.14 implies that

$$\mathcal{O} = \bigoplus_{\lambda \in \text{Irr}(W)} \mathcal{O}^{\bar{\mathbf{c}}\lambda}$$

with $\Delta(\lambda) \in \mathcal{O}^{\bar{\mathbf{c}}\lambda}$. Then exercise 3.13 implies that $\Delta(\lambda) = L(\lambda)$ and hence category \mathcal{O} is semi-simple. \square

Solution to exercise 2.26. (1) If $\mathbf{c} \notin \frac{1}{2} + \mathbb{Z}$ then category \mathcal{O} is semi-simple. The commutation relation of x and y implies that

$$[y, x^n] = \begin{cases} nx^{n-1} - 2\mathbf{c}x^{n-1}s & n \text{ odd} \\ nx^{n-1} & n \text{ even} \end{cases}$$

This implies that

$$L(\rho_0) = \frac{\mathbb{C}[x] \otimes \rho_0}{x^{2m+1}\mathbb{C}[x] \otimes \rho_0}, \quad L(\rho_1) = \Delta(\rho_1)$$

when $\mathbf{c} = \frac{1}{2} + m$ for some $m \in \mathbb{Z}_{\geq 0}$. Similarly,

$$L(\rho_0) = \Delta(\rho_0), \quad L(\rho_1) = \frac{\mathbb{C}[x] \otimes \rho_1}{x^{2m+1}\mathbb{C}[x] \otimes \rho_1}$$

when $\mathbf{c} = \frac{-1}{2} - m$ for some $m \in \mathbb{Z}_{\geq 0}$.

For part (2), we have $\mathbf{e}\mathbf{u} = xy - \mathbf{c}s$. Therefore $\mathbf{c}_0 = -\mathbf{c}$ and $\mathbf{c}_1 = \mathbf{c}$. Thus, $\rho_0 \leq_{\mathbf{c}} \rho_1$ if and only if $2\mathbf{c} \in \mathbb{Z}_{\geq 0}$. Similarly, $\rho_1 \leq_{\mathbf{c}} \rho_0$ if and only if $2\mathbf{c} \in \mathbb{Z}_{\leq 0}$. For all other \mathbf{c} , ρ_0 and ρ_1 are incomparable. \square

Solution to exercise 2.27. Let's consider the case $\mathbf{c} = \frac{1}{2} + m$ for some $m \in \mathbb{Z}_{\geq 0}$. The situation $\mathbf{c} = -\frac{1}{2} - m$ is completely analogous. BGG reciprocity implies that $\Delta(\rho_0) = P(\rho_0)$ and $P(\rho_1)$ is free of rank two over $\mathbb{C}[x]$. Moreover, we have a short exact sequence

$$0 \rightarrow \Delta(\rho_0) \rightarrow P(\rho_1) \rightarrow \Delta(\rho_1) = L(\rho_1) \rightarrow 0. \quad (15)$$

So, as graded \mathbb{Z}_2 -modules, we write $P(\rho_1) = \mathbb{C}[x] \otimes \rho_0 \oplus \mathbb{C}[x] \otimes \rho_1$, where $\mathbb{C}[x] \otimes \rho_0$ is identified with $\Delta(\rho_0)$. Then the structure of $P(\rho_1)$ is completely determined by the action of x and y on ρ_1 :

$$y \cdot (1 \otimes \rho_1) = f_1(x) \otimes \rho_0, \quad x \cdot (1 \otimes \rho_1) = x \otimes \rho_1 + f_0(x) \otimes \rho_0$$

for some $f_0, f_1 \in \mathbb{C}[x]$. For the action to be well-defined we must check the relation $[y, x] = 1 - 2\mathbf{c}s$, which reduces to the equation

$$y \cdot (f_0(x) \otimes \rho_0) = 0.$$

Also $s(f_i) = f_i$ for $i = 0, 1$. This implies that $f_0(x) = 1$. The second condition we require is that $P(\rho_1)$ is indecomposable (this will uniquely characterize $P(\rho_1)$ up to isomorphism). This is equivalent to asking that the short exact sequence (15) does not split. Choosing a splitting means choosing a vector $\rho_1 + f_2(x) \otimes \rho_0 \in P(\rho_1)$ such that $y \cdot (\rho_1 + f_2(x) \otimes \rho_0) = 0$. One can check that this is always possible, except when $f_1(x) = x^{2m}$. Thus we must take $f_1(x) = x^{2m}$. \square

Solution to exercise 2.28. We assume that $\mathbf{c} = \frac{1}{2} + m$ for some $m \in \mathbb{Z}_{\geq 0}$. The case $\mathbf{c} = -\frac{1}{2} - m$ is similar and all other cases are trivial. We need to describe $A = \text{End}_{\mathbf{H}_c(\mathbb{Z}_2)}(P(\rho_0) \oplus P(\rho_1))$. Using the general formula (prove it!) $\dim \text{Hom}_{\mathbf{H}_c(W)}(P(\lambda), M) = [M : L(\lambda)]$, and BGG reciprocity, we see that

$$\dim \text{End}_{\mathbf{H}_c(\mathbb{Z}_2)}(P(\rho_0)) = 1, \quad \dim \text{End}_{\mathbf{H}_c(\mathbb{Z}_2)}(P(\rho_1)) = 2 \quad (16)$$

$$\dim \text{Hom}_{\mathbf{H}_c(\mathbb{Z}_2)}(P(\rho_0), P(\rho_1)) = 1, \quad \dim \text{Hom}_{\mathbf{H}_c(\mathbb{Z}_2)}(P(\rho_1), P(\rho_0)) = 1. \quad (17)$$

Hence $\dim A = 5$. The algebra A will equal $\mathbb{C}Q/I$, where Q is some quiver and I an admissible ideal¹³. The vertices of Q are labeled by the simple modules in \mathcal{O} , hence there are two: e_0 and e_1 (corresponding to $L(\rho_0)$ and $L(\rho_1)$ respectively). The number of arrows from e_0 to e_1 equals $\dim \text{Ext}_{\mathbf{H}_c(\mathbb{Z}_2)}(L(\rho_0), L(\rho_1))$ and similarly for e_1 to e_0 . Hence there is one arrow $e_1 \leftarrow e_0 : a$ and one arrow $e_0 \leftarrow e_1 : b$. The projective module $P(\rho_0)$ will be a quotient of

$$\mathbb{C}Qe_0 = \mathbb{C}\{e_0, ae_0 = a, ba, aba, \dots\},$$

and similarly for $P(\rho_1)$. Equations (16) imply that $ba = (ab)^2 = 0$ in A (note that we cannot have $e_0 - \alpha ba = 0$ etc. because the endomorphism ring of an indecomposable is a local ring). Hence A is a quotient of $\mathbb{C}Q/I_0$, where $I_0 = \langle ba, (ab)^2 \rangle$. Now equation (17) implies that $bab = \alpha b$ for some $\alpha \in \mathbb{C}$. But $\alpha \neq 0$ implies that $ab = 0$ in A - a contradiction. Therefore $bab = 0$. \square

Solution to exercise 2.30. Part (1): By the proof of Corollary 1.17, we know that \mathbf{c} is aspherical if and only if

$$I := \mathbf{H}_c(W) \cdot \mathbf{e} \cdot \mathbf{H}_c(W)$$

is a proper two-sided ideal of $\mathbf{H}_c(W)$. If this is the case then there exists some primitive ideal J such that $I \subset J$. Hence Ginzburg's result implies that there is a simple module $L(\lambda)$ in category \mathcal{O} such that $I \cdot L(\lambda) = 0$. But clearly this happens if and only if $\mathbf{e} \cdot L(\lambda) = 0$.

Part (2): The only aspherical value for \mathbb{Z}_2 is $\mathbf{c} = -\frac{1}{2}$. \square

6.3. Solutions to exercises in lecture 3.

¹³Recall that an ideal I in a finite dimensional algebra B is said to be *admissible* if there exists some $m \geq 2$ such that $\text{rad}(B)^m \subset I \subset \text{rad}(B)^2$

Solution to exercise 3.8. The Young diagram with residues is

0						
1						
2	3	0				
3	0	1	2	3	0	
0	1	2	3	0	1	

Then $F_2|\lambda\rangle = q|(6, 6, 3, 2, 1)\rangle$, $K_1|\lambda\rangle = |\lambda\rangle$ and

$$E_4|\lambda\rangle = q^{-2}|(6, 5, 3, 1, 1)\rangle + q^{-1}|(6, 6, 2, 1, 1)\rangle + |(6, 6, 3, 1)\rangle.$$

□

Solution to exercise 3.10. By adding an infinite number of zeros to the end of $\lambda \in \mathcal{P}$, we may consider it as an infinite sequence $(\lambda_1, \lambda_2, \dots)$ with $\lambda_i \geq \lambda_{i+1}$ and $\lambda_N = 0$ for all i and all $N \gg 0$. Define $I(\lambda)$ by $i_k = \lambda_k + 1 - k$. It is clear that this rule defines a bijection with the required property. □

Solution to exercise 3.15. We have

$$\mathcal{G}([4]) = [4] + q[3, 1] + q[2, 1, 1] + q^2[1, 1, 1, 1], \quad \mathcal{G}([3, 1]) = [3, 1] + q[2, 2] + q^2[2, 1, 1],$$

$$\mathcal{G}([2, 2]) = [2, 2] + q[2, 1, 1], \quad \mathcal{G}([2, 1, 1]) = [2, 1, 1] + q[1, 1, 1, 1], \quad \mathcal{G}([1, 1, 1, 1]) = [1, 1, 1, 1].$$

and

$$\mathcal{G}([5]) = [5] + q[3, 1, 1] + q^2[1, 1, 1, 1, 1], \quad \mathcal{G}([4, 1]) = [4, 1] + q[2, 1, 1, 1],$$

$$\mathcal{G}([3, 2]) = [3, 2] + q[3, 1, 1] + q^2[2, 2, 1], \quad \mathcal{G}([2, 2, 1]) = [2, 2, 1],$$

$$\mathcal{G}([3, 1, 1]) = [3, 1, 1] + q[2, 2, 1] + q[1, 1, 1, 1, 1], \quad \mathcal{G}([2, 1, 1, 1]) = [2, 1, 1, 1],$$

$$\mathcal{G}([1, 1, 1, 1, 1]) = [1, 1, 1, 1, 1].$$

□

Solution to exercise 3.20. (1) In this case, the numbers $e_{\lambda, \mu}(1)$, \mathbf{c}_λ and $\dim \lambda$ are:

$\mu \backslash \lambda$	(5)	(4, 1)	(3, 2)	(3, 1, 1)	(2, 2, 1)	(2, 1, 1, 1)	(1, 1, 1, 1, 1)
(5)	1	0	0	0	0	0	0
(4, 1)	0	1	0	0	0	0	0
(3, 2)	0	0	1	0	0	0	0
(3, 1, 1)	-1	0	-1	1	0	0	0
(2, 2, 1)	1	0	0	-1	1	0	0
(2, 1, 1, 1)	0	-1	0	0	0	1	0
(1, 1, 1, 1, 1)	0	0	1	-1	0	0	1
\mathbf{c}_λ	$-\frac{65}{2}$	-15	$-\frac{9}{2}$	0	$\frac{19}{2}$	20	$\frac{75}{2}$
$\dim \lambda$	1	4	5	6	5	4	1

If we define $\text{ch}_\lambda(t) := (1-t)^5 \cdot \text{ch}(L(\lambda))$, then

$$\begin{aligned}\text{ch}_{(5)}(t) &= t^{-\frac{65}{2}} - 6t + 5t^{\frac{19}{2}}, & \text{ch}_{(4,1)}(t) &= 4t^{-15} - 4t^{20}, \\ \text{ch}_{(3,2)}(t) &= 5t^{-\frac{9}{2}} - 6t + t^{\frac{75}{2}}, & \text{ch}_{(3,1,1)}(t) &= 6t - 5t^{\frac{19}{2}} - t^{\frac{75}{2}}, \\ \text{ch}_{(2,2,1)}(t) &= 5t^{\frac{19}{2}}, & \text{ch}_{(2,1,1,1)}(t) &= 4t^{20}, & \text{ch}_{(1,1,1,1,1)}(t) &= t^{\frac{75}{2}}.\end{aligned}$$

(2) -

(3) It suffices to calculate the 2 (resp. 3 and 5) cores of the partitions of 5. For $r = 2$ we get

$$\{[5], [3, 2], [2, 2, 1], [1, 1, 1, 1, 1]\}, \quad \{[4, 1], [2, 1, 1, 1]\},$$

where the 2-cores are $[1]$ and $[2, 1]$ respectively. For $r = 3$ we get

$$\{[5], [2, 2, 1], [2, 1, 1, 1]\}, \quad \{[4, 1], [3, 2], [1, 1, 1, 1, 1]\},$$

where the 3-cores are $[2]$ and $[1, 1]$ respectively. For $r = 5$ we get

$$\{[5], [4, 1], [2, 1, 1, 1], [1, 1, 1, 1, 1]\}, \quad \{[3, 2]\}, \quad \{[2, 2, 1]\},$$

where the 5-cores are \emptyset , $[3, 2]$ and $[2, 2, 1]$ respectively (for an explanation of what is going on in this final example read [6]).

□

Solution to exercise 3.21. The argument is essentially identical to that for $\text{ch}(L(\lambda))$. We use the polynomials $e_{\lambda, \mu}(q)$ to express the character $\text{ch}_W(L(\lambda))$ in terms of the character of the standard modules $\Delta(\mu)$. Then, we just need to calculate $\text{ch}_W(\Delta(\lambda))$. Note that 1) $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ is a graded W -module 2) the **eu**-character of $\Delta(\lambda)$ is simply the graded character of $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ multiplied by $t^{\mathbf{e}\lambda}$. Hence, it suffices to describe $\mathbb{C}[\mathfrak{h}] \otimes \lambda$ as a graded W -module as a graded W -module. This factors as

$$\mathbb{C}[\mathfrak{h}] \otimes \lambda = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}]^{\text{co}W} \otimes \lambda$$

and hence

$$\text{ch}_W(\mathbb{C}[\mathfrak{h}] \otimes \lambda) = \frac{1}{\prod_{i=1}^n (1-t^i)} \text{ch}_W(\mathbb{C}[\mathfrak{h}]^{\text{co}W} \otimes \lambda).$$

We have

$$\text{ch}_W(\mathbb{C}[\mathfrak{h}]^{\text{co}W} \otimes \lambda) = \sum_{\mu \in \text{Irr}(W)} \left(\sum_{i \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}]_i^{\text{co}W} \otimes \lambda : \mu] t^i \right) [\mu] = \sum_{\mu \in \text{Irr}(W)} f_{\lambda, \mu}(t) [\mu].$$

In fact, for $W = \mathfrak{S}_4$, one can explicitly calculate the generalized fake polynomials.

$\mu \setminus \lambda$	(4)	(3, 1)	(2, 2)
(4)	1	$t + t^2 + t^3$	$t^2 + t^4$
(3, 1)	$t + t^2 + t^3$	$1 + t + 2t^2 + 2t^3 + 2t^4 + t^5$	$t + t^2 + 2t^3 + t^4 + t^5$
(2, 2)	$t^2 + t^4$	$t + t^2 + 2t^3 + t^4 + t^5$	$t + 2t^3 + t^5$
(2, 1, 1)	$t^3 + t^4 + t^5$	$t + 2t^2 + 2t^3 + 2t^4 + t^5 + t^6$	$1 + 2t^2 + t^3 + t^4 + t^5$
(1, 1, 1, 1)	t^6	$t^3 + t^4 + t^5$	$t + t^3$

$\mu \setminus \lambda$	$(2, 1, 1)$	$(1, 1, 1, 1)$
(4)	$t^3 + t^4 + t^5$	t^6
$(3, 1)$	$t + 2t^2 + 3t^3 + 2t^4 + t^5$	$t^3 + t^4 + t^5$
$(2, 2)$	$t + t^2 + 2t^3 + t^4 + t^5$	$t^2 + t^4$
$(2, 1, 1)$	$1 + t + 2t^2 + 2t^3 + 2t^4 + t^5$	$t + t^2 + t^3$
$(1, 1, 1, 1)$	$t + t^2 + t^3$	1

□

6.4. Solutions to exercises in lecture 4.

Solution to exercise 4.2. Let $z = x^2$ so that $\mathbb{C}[x]^{\mathbb{Z}_2} = \mathbb{C}[x^2] = \mathbb{C}[z]$. The ring $\mathcal{D}(\mathfrak{h})^W$ is generated by $x^2, x\partial_x$ and ∂_x^2 and $\mathcal{D}(\mathfrak{h}/W) = \mathbb{C}\langle z, \partial_z \rangle$. Since

$$x\partial_x(z^n) = x\partial_x(x^{2n}) = 2nx^{2n} = 2nz^n$$

and

$$\partial_x^2(z^n) = 2n(2n-1)z^{n-1}$$

we see that $\phi : \mathcal{D}(\mathfrak{h})^W \rightarrow \mathcal{D}(\mathfrak{h}/W)$ sends $x\partial_x$ to $2z\partial_z$ and ∂_x^2 is sent to $\partial_z(4z\partial_z - 2)$. Then ∂_z is not in the image of ϕ so it is not surjective. A rigorous way to show this is as follows: the morphism ϕ is filtered. Therefore, it induces a morphism on associated graded, this is the map

$$\begin{aligned} \mathbb{C}[A, B, C]/(AC - B^2) &\rightarrow \mathbb{C}[D, E], \\ A &\mapsto D, B \mapsto 2DE, C \mapsto 4DE^2. \end{aligned}$$

This is a proper embedding. □

Solution to exercise 4.9. Under the Dunkl embedding,

$$y = \partial_x - \frac{\mathbf{c}}{x}(1 - s),$$

which implies that $\partial_x \cdot rho_0 = 0$ and $\partial_x \cdot rho_1 = \frac{2\mathbf{c}}{x}e_1$. So the \mathbb{Z}_2 -equivariant local systems on \mathbb{C}^\times corresponding to $\Delta(\rho_0)[\delta^{-1}]$ and $\Delta(\rho_1)[\delta^{-1}]$ are given by the differential equations $\partial_x = 0$ and $\partial_x - \frac{2\mathbf{c}}{x} = 0$ respectively. Now we need to construct the corresponding local systems on $\mathbb{C}^\times/\mathbb{Z}_2$. If $z := x^2$ then $\partial_z = \frac{1}{2x}\partial_x$ and $\Delta(\rho_0)[\delta^{-1}]^{\mathbb{Z}_2}$, resp. $\Delta(\rho_1)[\delta^{-1}]^{\mathbb{Z}_2}$ has basis $a_0 = 1 \otimes \rho_0$, resp. $a_1 = x \otimes \rho_1$, as a free $\mathbb{C}[z^{\pm 1}]$ -module. We see that

$$\partial_z \cdot a_0 = 0, \quad \partial_z \cdot a_1 = \frac{1 + 2\mathbf{c}}{2z}a_1.$$

Since the solutions of these equations are 1 and $z^{\frac{1+2\mathbf{c}}{2}}$, the monodromy of these equations is given by $t \mapsto 1$ and $t \mapsto -\exp(2\pi\sqrt{-1}\mathbf{c}t)$ respectively. Therefore $\mathrm{KZ}(\Delta(\rho_0))$ is the one dimensional representation $\mathbb{C}b_0$ of $\mathrm{H}_{\mathbf{c}}(\mathbb{Z}_2)$ defined by $T \cdot b_0 = b_0$ and $\mathrm{KZ}(\Delta(\rho_1))$ is the one dimensional representation $\mathbb{C}b_1$ of $\mathrm{H}_{\mathbf{c}}(\mathbb{Z}_2)$ defined by $T \cdot b_1 = -\exp(2\pi\sqrt{-1}\mathbf{c})b_1$ respectively. □

Solution to exercise 4.10. Recall that $P(\rho_1)$ is the $\mathrm{H}_{\mathbf{c}}(\mathbb{Z}_2)$ -module $\mathbb{C}[x] \otimes \rho_1 \oplus \mathbb{C}[x] \otimes \rho_0$ with

$$x \cdot (1 \otimes \rho_1) = x \otimes \rho_1 + 1 \otimes \rho_0, \quad x \cdot (1 \otimes \rho_0) = x \otimes \rho_0,$$

$$y \cdot (1 \otimes \rho_1) = x^{2m} \otimes \rho_0, \quad y \cdot (1 \otimes \rho_0) = 0.$$

One can check that this implies that

$$\frac{1}{x} \cdot (1 \otimes \rho_1) = \frac{1}{x} \otimes \rho_1 - \frac{1}{x^2} \otimes \rho_0.$$

If we write $P(\rho_0)[\delta^{-1}] = \mathbb{C}[x^{\pm 1}] \cdot a_1 \oplus \mathbb{C}[x^{\pm 1}] \cdot a_0$, where $a_1 = 1 \otimes \rho_1$ and $a_0 = 1 \otimes \rho_0$, then

$$\partial_x \cdot a_1 = (y + \frac{c}{x}(1-s)) \cdot a_1 = y \cdot a_1 + \frac{2c}{x} \cdot a_1.$$

Now,

$$y \cdot a_1 = x^{2m} \otimes \rho_0 = x^{2m} \cdot a_0,$$

hence $\partial_x \cdot a_1 = \frac{2c}{x} \cdot a_1 + x^{2m} \cdot a_0$. Also, $\partial_x \cdot a_0 = 0$. A free $\mathbb{C}[z^{\pm 1}]$ -basis of $P(\rho_0)[\delta^{-1}]^{\mathbb{Z}_2}$ is given by $u_1 = x \cdot a_1$ and $u_0 = a_0$. Therefore

$$\partial_z \cdot u_1 = \frac{1+2c}{2z} u_1 + \frac{1}{2} z^m u_0 = \frac{m+1}{z} u_1 + \frac{1}{2} z^m u_0$$

and $\partial_z \cdot u_0 = 0$, where we used the fact that $\mathbf{c} = \frac{1}{2} + m$. Hence $\text{KZ}(P(\rho_1))$ is given by the connection

$$\partial_z + \begin{pmatrix} \frac{m+1}{z} & 0 \\ \frac{1}{2} z^m & 0 \end{pmatrix}.$$

Two linearly independent solutions of this equation are

$$g_1(z) = \begin{pmatrix} z^{-(m+1)} \\ \frac{1}{2} \ln(z) \end{pmatrix}, \quad g_2(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If, in a small, simply connected neighbourhood of 1, we choose the branch of $\ln(z)$ such that $\ln(1) = 0$, then $\gamma(0) = 0$ and $\gamma(1) = 2\pi\sqrt{-1}$, where

$$\gamma : [0, 1] \rightarrow \mathbb{C}, \quad \gamma(t) = \ln(\exp(2\pi\sqrt{-1}t)).$$

Therefore $\text{KZ}(P(\rho_1))$ is the two-dimensional representation of $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_2)$ given by

$$T \mapsto \begin{pmatrix} 1 & 0 \\ 2\pi\sqrt{-1} & 1 \end{pmatrix}.$$

This is isomorphic to the left regular representation of $\mathcal{H}_{\mathbf{q}}(\mathbb{Z}_2)$. □

Solution to exercise 4.11. If $\mathbf{c} = \frac{1}{2} + m$ for some $m \in \mathbb{Z}_{\geq 0}$ then $P_{KZ} = P(\rho_1)$. If $\mathbf{c} = -\frac{1}{2} - m$ for some $m \in \mathbb{Z}_{\geq 0}$ then $P_{KZ} = P(\rho_0)$. Otherwise, $P_{KZ} = P(\rho_0) \oplus P(\rho_1)$, □

6.5. Solutions to exercises in lecture 5.

Solution to exercise 5.10. Part (1): Recall that Hayashi's construction, [34] is defined as follows. We may think of t as a variable so that $\mathbf{H}_{0,\mathbf{c}}(G) = \mathbf{H}_{t,\mathbf{c}}(G)/t \cdot \mathbf{H}_{t,\mathbf{c}}(G)$. For $z_1, z_2 \in \mathbf{Z}_{\mathbf{c}}(G)$ define

$$\{z_1, z_2\} = \left(\frac{1}{t} [\hat{z}_1, \hat{z}_2] \right) \bmod t\mathbf{H}_{t,\mathbf{c}}(G),$$

where \hat{z}_1, \hat{z}_2 are arbitrary lifts of z_1, z_2 in $\mathbf{H}_{t,\mathbf{c}}(G)$. Write $\rho : \mathbf{H}_{t,\mathbf{c}}(G) \rightarrow \mathbf{H}_{0,\mathbf{c}}(G)$ for the quotient map. Let us first check that the binary operation is well-defined. Let \hat{z}_1, \hat{z}_2 be arbitrary lifts of

$z_1, z_2 \in Z_{\mathbf{c}}(G)$. Then $\rho([\hat{z}_1, \hat{z}_2]) = [\rho(\hat{z}_1), \rho(\hat{z}_2)] = 0$. Therefore, there exists some $\hat{z}_3 \in H_{t, \mathbf{c}}(G)$ such that $[\hat{z}_1, \hat{z}_2] = t \cdot \hat{z}_3$. Since t is a non-zero divisor, \hat{z}_3 is uniquely define. The claim is that $\rho(\hat{z}_3) \in Z_{\mathbf{c}}(G)$: let $h \in H_{0, \mathbf{c}}(G)$ and \hat{h} an arbitrary lift of h in $H_{t, \mathbf{c}}(G)$. Then

$$\begin{aligned} [h, \rho(\hat{z}_3)] &= \rho([\hat{h}, \hat{z}_3]) = \rho([\hat{h}, \frac{1}{t}[\hat{z}_1, \hat{z}_2]]) \\ &= -\rho(\frac{1}{t}[\hat{z}_2, [\hat{h}, \hat{z}_1]]) - \rho(\frac{1}{t}[\hat{z}_2, [\hat{h}, \hat{z}_1]]). \end{aligned}$$

Since \hat{z}_1 and \hat{z}_1 are lifts of central elements, the expressions $[\hat{z}_2, [\hat{h}, \hat{z}_1]]$ and $[\hat{z}_2, [\hat{h}, \hat{z}_1]]$ are in $t^2 H_{t, \mathbf{c}}(G)$. Hence $[h, \rho(\hat{z}_3)] = 0$. Therefore, the expression $\{z_1, z_2\}$ is well-defined. The fact that $\rho([t \cdot \hat{z}_1, \hat{z}_2]/t) = [\rho(\hat{z}_1), \rho(\hat{z}_2)] = 0$ implies that the bracket is independent of choice of lifts. The fact that the bracket makes $Z_{\mathbf{c}}(G)$ into a Lie algebra and satisfies the derivation property is a consequence of the fact that the commutator bracket of an algebra also has these properties.

Part (2) is left to the reader. \square

Solution to exercise 5.12. (1) Recall that each function $f \in \mathbb{C}[x, y]$ defines a vector field $\{f, -\}$ on \mathbb{C}^2 . For the generators x, y , these vector fields are $\{x, -\} = y\partial_y$, $\{y, -\} = -y\partial_x$. The idea is to calculate the integral curve through a point $(p, q) \in \mathbb{C}^2$ for each of these vector fields. Then the leaf through (p, q) will be the submanifold traced out by all these curves. We begin with $y\partial_y$. The corresponding integral curve is $a = (a_1(t), a_2(t)) : B_{\epsilon}(0) \rightarrow \mathbb{C}^2$ such that $a(0) = (p, q)$ and

$$a'(t) = (y\partial_y)_{a(t)}, \quad \forall t \in B_{\epsilon}(0).$$

Thus, $a'_1(t) = 0$ and $a'_2(t) = a_2(t)$ which means $a = (p, qe^t)$. Similarly, if $b = (b_1, b_2)$ is the integral curve through (p, q) for $-y\partial_x$ then $b = (-qt + p, q)$. Therefore, there are only two symplectic leaves, $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$.

(2) The Hamiltonian vector fields in this case are $X_E = H\partial_F - 2E\partial_H$, $X_F = -H\partial_E + 2F\partial_H$ and $X_H = 2E\partial_E - 2F\partial_F$. Let $a = (a_E(t), a_F(t), a_H(t))$ be an integral curve through (p, q, r) for a vector field X .

- For X_E , $a(t) = (p, -pt^2 + rt + q, -2pt + r)$.
- For X_F , $a(t) = (-qt^2 - rt + p, q, 2qt + r)$.
- For X_H , $a(t) = (p \exp(2t), q \exp(-2t), r)$.

Then, for all $(p, q, r) \neq (0, 0, 0)$, X_E, X_F, X_H span a two-dimensional subspace of $T_{(p, q, r)} \mathfrak{sl}_2^*$. Also, one can check that the expression $a_E(t)a_F(t) + \frac{1}{4}a_H(t)^2 = pq + \frac{1}{4}r^2$ is independent of t for each of the three integral curves above e.g. for X_F we have

$$(-qt^2 - rt + p)q + \frac{1}{4}(2qt + r)^2 = pq + \frac{1}{4}r^2.$$

Therefore, for $s \neq 0$, $V(EF + \frac{1}{4}H^2 = s)$ is a smooth, two-dimensional Poisson subvariety on which the symplectic form is everywhere non-degenerate. This implies that it is a symplectic leaf. For the null cone $\mathcal{N} = V(EF + \frac{1}{4}H^2 = 0)$, if we consider $U = \mathcal{N} \setminus \{(0, 0, 0)\}$, then this is also smooth and the symplectic form is everywhere non-degenerate. Thus, it is certainly

contained in a symplectic leaf of \mathcal{N} . However, the whole of \mathcal{N} cannot be a leaf because it is singular. Therefore, the symplectic leaves of \mathcal{N} are U and $\{(0, 0, 0)\}$. \square

Solution to exercise 5.24. Part (1): The fact that $H_{t,c}(W)$ is \mathbb{Z} -graded follows directly from the defining relations. The key thing to check is that the relations (2) are all homogeneous. The restricted rational Cherednik algebra $\overline{H}_c(W)$ is the quotient of $H_c(W)$ by the ideal I generated by $\mathbb{C}[\mathfrak{h}]_+^W$ and $\mathbb{C}[\mathfrak{h}^*]_+^W$. Both of these are graded subspaces of $H_c(W)$. Hence I is also graded. This implies that the quotient $\overline{H}_c(W)$ is graded.

Let $\lambda \in \text{Irr}(W)$. Recall that $\Delta(\lambda) = \overline{H}_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \lambda$, where $\mathbb{C}[\mathfrak{h}^*]_+^{coW}$ acts trivially on λ . If we put λ in degree zero, then it is a graded $\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W$ -module. This implies that the induced module is also graded. Since $1 \otimes \lambda$ generates $\Delta(\lambda)$ and is an irreducible, homogeneous space, there is a unique maximal graded proper submodule of $\Delta(\lambda)$, such that the corresponding quotient is $L(\lambda)$.

Part (2): Recall that Müller's Theorem from section 5.6 says that the primitive central idempotents of $\overline{H}_c(W)$ (the b_i 's) are the images of the primitive idempotents of $Z_c/A_+ \cdot Z_c$ under the natural map $Z_c/A_+ \cdot Z_c \rightarrow \overline{H}_c(W)$. The primitive idempotents of $R := Z_c/A_+ \cdot Z_c$ are in bijection with the maximal ideals in this ring: given a maximal ideal \mathfrak{m} in R there is a unique primitive idempotent b whose image in R/\mathfrak{m} is non-zero. Under this correspondence, the simple module belonging to the block of $\overline{H}_c(W)$ corresponding to b are precisely those simple modules supported at \mathfrak{m} . Thus, the block b has just one simple module if and only if there is a unique simple module of $H_c(W)$ supported at \mathfrak{m} . But, by the Artin-Procesi Theorem, remark 5.4 (3), there is a unique simple module supported at \mathfrak{m} if and only if \mathfrak{m} is in the Azumaya locus of $X_c(W)$. As noted in the proof of Corollary 5.6, the smooth locus of $X_c(W)$ equals the Azumaya locus. Thus, to summarize, the module $L(\lambda)$ is on its own in a block if and only if its support is contained in the smooth locus. Then, the conclusions of the exercise follow from Theorem 5.5 and Corollary 5.6. \square

Solution to exercise 5.25. The fact that $\Delta(\lambda)$ is indecomposable implies that all its composition factors belong to the same block. We know that $L(\lambda)$ is one of these composition factors. But, by assumption, $L(\lambda)$ is on its own in a block. Therefore every composition factor is isomorphic to $L(\lambda)$. So let's try and calculate the graded multiplicities of $L(\lambda)$ in $\Delta(\lambda)$. In the graded Grothendieck group of $\overline{H}_c(W)$, we must have

$$[\Delta(\lambda)] = [L(\lambda)][i_1] + \cdots + [L(\lambda)][i_r] \quad (18)$$

where $[L(\lambda)][k]$ denotes the class of $L(\lambda)$, shifted in degree by k . By exercise 5.24, $\dim L(\lambda) = |W|$ and it is easy to see that $\dim \Delta(\lambda) = |W| \dim \lambda$. Thus, $r = \dim \lambda$. Since $L(\lambda)$ is a graded quotient of $\Delta(\lambda)$ we may also assume that $i_1 = 0$. Recall that $\mathbb{C}[\mathfrak{h}]^{coW}$ is isomorphic to the regular representation as a W -module. Therefore, the fact that $[\mu \otimes \lambda : \text{triv}] \neq 0$ if and only if $\mu \simeq \lambda^*$ (in which case it is one) implies that the multiplicity space of the trivial representation in $\Delta(\lambda)$ is $(\dim \lambda)$ -dimensional. The Poincaré polynomial of this multiplicity space is precisely the fake polynomial $f_{\lambda^*}(t)$. On the other hand, the trivial representation only occurs once in $L(\lambda)$ since it is isomorphic to the regular representation. Therefore, comparing graded multiplicities of the trivial

representation on both sides of equation (18) implies that, up to a shift, $t^{i_1} + \cdots + t^{i_r} = f_{\lambda^*}(t)$. What is the shift? Well, the lowest degree¹⁴ occurring in $t^{i_1} + \cdots + t^{i_r}$ is $i_1 = 0$. But the lowest degree in $f_{\lambda}(t)$ is b_{λ^*} . Thus, $t^{i_1} + \cdots + t^{i_r} = t^{-b_{\lambda^*}} f_{\lambda^*}(t)$. This implies that

$$P(L(\lambda), t) = \frac{t^{b_{\lambda^*}} P(\Delta(\lambda), t)}{f_{\lambda^*}(t)}.$$

Clearly, $P(\Delta(\lambda), t) = (\dim \lambda) P(\mathbb{C}[\mathfrak{h}]^{\text{co}W}, t)$. □

Solution to exercise 5.26. The space $X_{\mathbf{c}}(G_2)$ is never smooth. The Poincaré polynomial of $\mathbb{C}[\mathfrak{h}_1]^{\text{co}G_2}$ is

$$(1 - t^2)(1 - t^6)/(1 - t)^2 = 1 + 2t + 2t^2 + \cdots + 2t^5 + t^6.$$

The polynomials $t^2 + t^4$ and $t + t^5$ do not divide this polynomial in $\mathbb{Z}[t, t^{-1}]$. Therefore

$$\dim L(\mathfrak{h}_1), \dim L(\mathfrak{h}_2) < 12$$

for any parameter \mathbf{c} , which implies the claim. □

Solution to exercise 5.27. For help with this exercise, read [3]. □

¹⁴It is not complete obvious that 0 is the lowest degree in $t^{i_1} + \cdots + t^{i_r}$, see [29, Lemma 4.4].

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